16. *Fundamental Theory of Toothed Gearing (I).*

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Suppose that two plane curves K_1 and K_2 roll without sliding mutually one along the other, and let F_1 and F_2 which roll, in accordance with the rolling motion of K_1 and K_2, with sliding mutually one along the other be the two plane curves invariably connected with K_1 and K_2 respectively. K_1 and K_2 are called the corresponding pitch curves, and F_1 and F_2 the corresponding profile curves. Furthermore, we shall call any two points of K_1 and K_2 which may fall on each other at the rolling motion of K_1 and K_2 the corresponding pitch points, and especially a point at which K_1 and K_2 are touching at a certain instant the common pitch point at the instant.

From now on we confine ourselves to deal with such continuous (pitch or profile) curves as at each of points on them a single tangent may be drawn continuously (although cusps are allowed to exist), and suppose that they touch each other always at one point during the motion.

§ 1. Necessary and sufficient conditions for profile curves (1).

As a necessary condition that two curves F_1 and F_2 invariably connected with two pitch curves K_1 and K_2 respectively be a pair of profile curves the following Descartes' theorem (a) is well known.

(a) *The common normal to the curves F_1 and F_2 at any point of contact of them always passes through a common pitch point*

From the condition (a) we obtain the following necessary and sufficient condition for profile curves.

Theorem 1. *A necessary and sufficient condition that two curves F_1 and F_2 invariably connected with two pitch curves K_1 and K_2 respectively be a pair of profile curves is that two perpendiculars from any common pitch point to F_1 and F_2 coincide with each other in the direction and in the length to their feet.*

As a necessary condition that a curve F settled at one K of pitch curves K_1 and K_2 be a profile curve, that is, there exists a corresponding curve to F which makes sliding contact motion with F, we can derive directly the following condition (b) from the condition (a).

(b) *The curve F is an envelope of a family of circles, each of which* *has its center on the curve K and touch F at one point.*

We can not assure that F has, in this case, no common point other than the point of contact with any circle of the family. Further it should be remarked that there appear two envelopes of a family of circles with centers on a curve K, if they exist, and that two feet of perpendiculars drawn from an arbitrary point P on K to these two envelopes take symmetric positions as regards the tangent to K at P.

Now we shall say two families of circles are developable from one upon another, if they consist of circles having centers at corresponding pitch points on K_1 and K_2 and equal radii.

From the condition (b) we have the following necessary and sufficient condition:

Theorem 2. *A necessary and sufficient condition that two curves F_1 and F_2 invariably connected with two pitch curves K_1 and K_2 respectively be a pair of profile curves is that they be a pair of suitably chosen envelopes of two families of circles being developable upon each other having centers on the curves K_1 and K_2.*

Proof of sufficiency. From the envelopes settled at K_1 and K_2 we shall choose and assort such two those as two feet of perpendicular drawn from each of common pitch points P to the envelopes exist on the same side of the common tangent of K_1 and K_2 at the point P, and denote them by F_1 and F_2. It is evident that the perpendiculars from the point P to such curves F_1 and F_2 have equal lengths. Further their directions coincide likewise as we shall verify in the following report (II), § 1.

From Theorem 2 we can easily derive the following theorem concerning the interchangeability of profile curves.

Given three curves K_r, K_1 and K_2 which are all touching at the same one point and starting from this position may roll without sliding along one another. Let F_r and F_1 be a pair of profile curves invariably connected with the pitch curves K_r and K_1, and similarly F_r and F_2 with K_r and K_2. Then F_1 and F_2 are a pair of profile curves having K_1 and K_2 as a pair of pitch curves.

As a special case of this theorem we obtain the the following Camus', when the curve F_r is reduced to a point C.

Given three curves K_r, K_1 and K_2 which are all touching at the same one point and starting from this position may roll without sliding along one an *other. Let* F_1 *and* F_2 *be the roulettes drawn by the same one point* C *invariably connected with the curve* K_r *when* K_r *makes rolling contact motion along* K_1 *and* K_2 *respectively. Then the curves* F_1 *and* F_2 *are a pair of profile curves having* K_1 *and* K_2 *as a pair of pitch curves.*

In addition, in this case, assume in particular K_1 be a circle and K_r be a circle with a radius half of K_1's, and choose the drawing point C on the perimeter of K_r then the roulette F_1 becomes a diameter of K_1. Accordingly we have the ollowing Chasles' theorem:

The curve generated by a diameter of a circle K_1 *during rolling contact motion of* K_1 *along any curve* K_2 *coincides with the roulette drawn by a fixed point on a circle* K_r *with a radius half of* K_1*'s when* K_r *makes rolling contact motion along* K_2.

Now we may consider the condition (b) as a mere condition for a curve F settled at a pitch curve K. From the condition (b) we can derive the following condition:

(c) *Two normals of the curve* F *at any two points on* F *do not pass through the same pitch point.*

Accordingly, to any respective point on F there corresponds one pitch point on K.

The condition (c) is obviously equivalent to the following condition:

(d) *When a point runs on the curve* F *to a certain direction the pitch point corresponding to it runs on the curve* K *also to a definite direction.*

At this time there may take place one of the following three cases, namely, the case when the normals drawn at any two points of F intersect by no means before they arrive at the pitch points corresponding to those, or the case when they always intersect, or the case different from the former two. We shall say that the curve F is of positive type in the first case, negative type in the second case, and monotype in general term for these two types. In the last case we may consider the curve F consisting of several monotypical curves and we shall say that F is of mixed type.

Now suppose that the curves K and F satisfying the relation of the condition (d) are given. Take the points $C, C', C'', \ldots\ldots, C^{(n)}, C^{(n+1)}, \ldots\ldots$ starting from an arbitrary point C on the curve F to a definite direction one after another, then the points $P, P', P'', \ldots\ldots, P^{(n)}, P^{(n+1)}, \ldots\ldots$ on the curve K which run also to a definite direction are defined in correspondence to those. Next, we shall rotate the given figure at any point $P^{(n)}$ as a center until the straight line $P^{(n)}\,C^{(n+1)}$ is reached to the original position of the straight line

$P^{(n)} C^{(n)}$, and denote by $Q'^{n+1)}$ the position taken by the point $P^{(n+1)}$. By means of this operation there correspond the points Q', Q'', Q''',......to the points P', P'', P''', ... respectively. After returning the figure to the original position we shall rotate it again at the point P as a center and put the point P' upon the point Q' denoting it newly by R', and denote by R'' the position taken at this time by the point Q''. And next, starting from this state we rotate the figure at the point R' as a center until the point P'' is put upon the point R'' and denote by R''' the position taken at this time by the point Q'''. Repeating this operation succesively we get a polygonal line $P R' R''$...... starting from the point P. The circumstance is the same when we set the dividing points of F in the opposite direction starting from C. When we increase indefinitely the number of the dividing points of F and let the distance of any two adjacent points tend to zero, we may have a limiting curve K_r touching the curve K at the point P. The curve F is given as a roulette drawn by the point C when the curve K_r rolls along the curve K without sliding. Thus we find the following property (e) of the curve satisfying the condition (d).

(e) *The curve F is a roulette drawn by a rolling curve and a drawing point suitably taken using the curve K as a base curve.*

It should not be difficult to understand that from the condition (e) for the curve F follows the condition (b), consequently the four conditions (b), (c), (d) and (e) are all equivalent to one another. Moreover, when we roll the curve K_r defined by the condition (e) along the mate of the pitch curve K without sliding we get one more roulette drawn by the same drawing point C. This roulette and the curve F are a pair of profile curves by the Camus' theorem Hence the condition (e) is a necessary and sufficient condition that the curve F is a profile curve and consequently each of the conditions (b), (c) and (d) equivalent to (e) is so. Thus we have

Theorem 3. *A necessary and sufficient condition that a curve F invariably connected with one K of pitch curves K_1 and K_2 be a profile curve is* (b): *the curve F is an envelope of a family of circles, each of which has its center on the curve K and touch F at one point, or* (c): *two normals of the curve F at any two points on it do not pass through the same pitch point, or* (d): *when a point runs on the curve F to a certain direction, the pitch point corresponding to it runs on the curve K also to a definite direction, or* (e): *the curve F is a roulette drawn by a rolling curve and a drawing point suitably defined using K as a base curve.*

It is not necessary to give the notice that each of the given profile curves F_1

and F_2 composing of a pair is a roulette drawn by the same rolling curve and drawing point with the pitch curves K_1 and K_2 as the base curves respectively.

In conclusion we shall consider a parallel curve F^* of a given profile curve F. For example, the condition (d) is evidently satisfied for the curve F^* as for the original curve F, and so by Theorem 3 we get the following theorem:

Any parallel curve of a given profile curve is also a profile curve for the original pitch curve.

§ 2. Necessary and sufficient condition for profile curves (2).

In the case that the curve F is particularly of monotype we can derive the following sharper property (b)* of F than (b).

(b)* *The curve F is such an envelope of a family of circles with centers on the curve K as each of the circles touch F at one point and has no common point with F except the point of contact.*

Proof. Let S be a circle of the given family, P be the center of S, and C be the point of contact of S and F. Now suppose that F and S have common points other than C, and denote the nearest point of them to C on the curve F by D. As the curve F is of monotype, there exists no cusp between the points C and D. And then there appears one of the three cases that D is not an end point of F but F and S touch at D, or F and S intersect at D, or D is an end point of F and falls just on S. In the first case the normal of F at this point D passes through the pitch point P on the contrary of the property (c) of F. In the second case we denote the points of intersection of a half-straight line starting from P and the curves F and S by X and Y respectively, and the angle between the tangents to F and S at these points respectively by τ. The value of τ varies continuously. At the point C $\tau=0$ and in the neighbourhood on the D point side of the point C $\tau>0$ or $\tau<0$ and at D $\tau<0$ or $\tau>0$ correspondingly. Therefore we have at least one point X between C and D on F at which point $\tau=0$. At such point X the normal to F passes again through the point P. In the last case when D is an end point of F it leads to the same contradiction. Hence the condition (b)* must be satisfied.

It is no doubt that the condition (b) should be derived from the condition (b)*. Moreover, of course, a curve F having the property of the condition (b)* is of monotype. Thus we obtain the following:

Theorem 4. *A necessary and sufficient condition that a curve F invariably connected with the pitch curve K be a profile curve of monotype is that F is such an envelope of a family of circles with centers on K as each of the* *circles touchs F at one point and has no common point with F except the point of contact.*

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