23. *On the Potential Defined in a Domain.*

By Tadao KUBO.
Tokushima Technical College.

(Comm. by T. TAKAGI, M. J. A., April 12, 1949.)

Let us consider a simply connected "schlicht" domain R on the z-plane whose boundary is a simple closed Jordan curve and an additive class F composed of the sets of points contained in a bounded closed subset E of R.

We suppose that a function μ ($\geqq 0$) of the sets is completely additive with respect to any set belonging to F.

Then we shall define the potential of mass-distribution μ on E in the form

$$V(z)=\int_E g(z,\zeta)d\mu(\zeta), \tag{1}$$

where g (z, ζ) is a Green's function of the domain R with a pole ζ and z is any fixed point in R.

The integral(1) has a meaning in the sense of the Stieltjes Lebesgue-Radon's integral.

From the definition(1), we easily obtain

$$\Delta V(z)=0 \qquad (\Delta \text{ is Laplacian})$$

at any point in the free space R-E, for $\Delta g(z,\zeta)=0$.

Now we shall study whether Gauss' theorems[1] on the potential in the usual sense hold for the potentil (1) in our definition, succeeding to the idea of "Green's Geometry"[2] discussed by Prof. Matsumoto.

Let the subset E be lying entirely in R. Then we can suitably choose a constant c (>0) such that the subset E is entirely enclosed by the equipotential curve C_0 : $g(z,z_0)=c$ of Green's function of R with a pole z_0

Thus, let us consider the arithmetic mean of the potential (1) by integration on C_0 for which we shall use the non-Euclidean (hyperbolic) metric $d\sigma_z$[3] for the linear element.

Such an arithmetic mean by integration, we denote by $A\{V_{(z)}\}$ for simplicity.

By Fubini's theorem on the change of order of integration, we have

$$\int_{C_0} V(z)\,d\sigma_z=\int_E\left(\int_{C_0} g(z,\zeta)d\sigma_z\right)d\mu(\zeta) \tag{2}$$

---

1) O. D. Kellogg : Foundations of Potential Theory (1929) P. 82.
2) T. Matsumoto : Gekkan 'Sugaku' October, November, (1937).
3) R. Nevanlinna : Eindeutige Analytische Funktionen (1936) S. 48.

Next we represent conformally the domain R on the unit circle K on the x-plane by a regular function $x=x(z)$ such that the pole z is carried into the center of K.

In this representation, let any one point ς in E correspond to a point ξ in the circle K.

Then, it follows that if we denote the inverse function of $x(z)$ by $z(x)$, $g(z(x), \varsigma)$ is a Green's function of the circle K on the x-plane with a pole ξ. And if we denote the function by $g(x, \xi)$, we have

$$(3) \qquad g(x,\xi)=\log\left|\frac{1-\bar{\xi}x}{x-\xi}\right| = \log\frac{1}{|x-\xi|}+\log|1-\bar{\xi}x|.$$

Let us transform the integral $\int_{C_0} g(z, \varsigma)d\sigma_z$ in (2) into the integral in the x-plane by above conformal transformation under which the hyperbolic linearelement is invariant and

$$(4) \qquad d\sigma_z=d\sigma_x=\frac{ds_x}{1-|x|^2}$$

where ds_x is the linearelement in the usual sense.

Here, the equipotential curve C_0 is transformed into a circle K_0 on the x-plane whose center is the origin and whose radius $\rho=\exp(-c)$.

Accordingly we have by (3) and (4)

$$\int_{C_0} g(z,\varsigma)\,d\sigma_z=\int_{K_0} g(x,\xi)\,d\sigma_x = \frac{1}{1-\rho^2}\left(\int_{K_0}\log\frac{1}{|x-\xi|}ds_x+\int_{K_0}\log|1-\bar{\xi}x|\,ds_x\right).$$

By elmentary reckoning,

$$(5) \qquad \frac{1}{2\pi\rho}\int_{K_0}\log\frac{1}{|x-\xi|}ds_x=\log\frac{1}{\rho}.$$

And by the mean-valued theorem of the harmonic function,

$$(6) \qquad \frac{1}{2\pi\rho}\int_{K_0}\log|1-\bar{\xi}x|\,ds_x=\log 1=0.$$

Therefore, we have by (5) and (6)

$$\frac{1-\rho^2}{2\pi\rho}\int_{K_0} g(x,\xi)d\sigma_x=\log\frac{1}{\rho}=c,$$

where $\frac{2\pi\rho}{1-\rho^2}$ is the hyperbolic length of the circumference of K_0 and also of C_0.

By above result and (2), it can be proved that

$$(7) \qquad A\{V(z)\}=c\int_E d\mu(\varsigma).$$

Since $\int_E d\mu(\varsigma)$ in (7) is the total mass of E, the following theorem is established.

Theorem I. *The average on the circumference of a non-Euclidean circle $g(z, z_0)=c$ of the potential.* (1) *of masses lying entirely inside of the cirele is independent of their distfibution within the circle, and is equal to their total mass divided by the constant* $1/c$.

Here the constant $1/c$ can be regarded as the non-Euclidean radius of the circle.

Moreover we have the following theorem similarly with above.

Theorem II. *The average on the circumference of a non-Euclidean circle $g(z, z_0)=c$ of the circumference af* (1) *of masses lying entirely outside of the circle is equal to the value of the potential at the center z_0.*

To get the last theorem, we have only to substitute

$$(5') \quad \frac{1}{2\pi\rho}\int_{K_0}\log\frac{1}{|x-\xi|}ds_x=\log\frac{1}{\rho'} \quad [\rho'=|\xi|=\exp(-g(\zeta,z_0))]$$
$$=g(\varsigma,z\ )=g(z_0,\varsigma)$$

for (5)

Thus we have

$$(7') \qquad A\{V(z)\}=\int_E g(z_0,\varsigma)\ d\mu(\varsigma)=V(z_0).$$