54. Supplementary Remarks on Frobeniusean Algebras. I.

By Tadasi NAKAYAMA.

Department of Mathematics, Nagoya University.

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The purpose of the present note is to make a supplementary remark to our previous structural criteria for Frobeniusean and quasi-Frobeniusean algebras.[1] The criteria were used to extend the notions to general rings with minimum condition, but our present remark is primarily concerned with the case of algebras[2] ratheh than rings with minimum condition. We show namely that in eacr of the criteria one half may be dropped in case of algebras. The result is applied to refine our previous result on residue-algebras. It may be used also to show that our previous criteria of Frobeniusean and quasi-Frobeniusean algebras in terms of duality of annihilation ideals may be reduced to one half too (in case of algebras, not rings with minimum condition);[3] this, together with some other results, we shall reserve for a succeeding joint note by M. Ikeda and myself.

Theorem 1. *Let A be an algebra over a field F, and N be its radical; the left annihilator $R = l(N)$ of N being the largest fully reducible right ideal of A. Suppose, now, that A possesses a left unit element E and that each right ideal $e_\kappa R$ is irreducible and isomorphic to $e_{\pi(\kappa)} A/e_{\pi(\kappa)} N$ where $\{e_\kappa(\kappa = 1, 2, \ldots, k)\}$ is a maximal*

1) An algebra is called Frobeniusean if it possesses unit element and if its left and right regular representations are equivalent. It is called quasi-Frobeniusean if it possesses unit element and if the totalities of distinct components in its left and right regular representations coincide. For their structure cf. T. Nakayama, On Frobeniusean algebras, I, Ann. Math. 40 (1941); III, Jap. Journ. Math. 18 (1942). — The alluded criteria were given in I, § 2, Lemma 2; cf. also II, § 4.

The writer wants to take this opportunity to make up a miss in III, which Mr. O. Nagai has kindly pointed out to the writer. Namely, in the Corollary to Theorem 1 of the paper we have to make assumption of the existence of a unit element (or else a some assumption which assures that the radical be contained in every maximal left or right ideal.

2) Under an algebra we always mean an algebra with finite rank (over a ground field).

3) The present study has been given impulsion by this problem raised by M. Ikeda.

set of mutually non-isomorphic[4] *primitive idempotent elements in* A *and* $\kappa \to \pi(\kappa)$ *is a permutation of* $(1, 2, \ldots, k)$. *Then the (right-left) symmetry counter-part prevails too and thus, in virtue of our previous result)* A *is quasi-Frobeninsean. If further the* κ*-capacity*[5] $f(\kappa) \leqq$ *the* $\pi(\kappa)$*-capacity* $f(\pi(\kappa))$, *for every* κ, *then* A *is Frobeniusean.*[6]

Proof. Let

$$E = \sum e_{\kappa,i}$$

be a decomposition of E into mutually orthogonal idempotent elements $e_{\kappa,i}(\kappa = 1, 2, \ldots, k;\ i = 1, 2, \ldots, f(\kappa))$, such that $e_{\kappa,i}$ are isomorphic to e_κ. Put $E_\kappa = \sum_i e_{\kappa,i}$.

$$R = ER = \sum E_\kappa R = \sum e_{\kappa,i} R,$$

the summations being direct. Each $e_{\kappa,i}R$ is isomorphic to $\bar{e}_{\pi(\kappa)}\bar{A}$; bar indicating "mod N". As $\bar{e}_\lambda \bar{A} \bar{E}_\lambda = \bar{e}_\lambda \bar{A}$, we have $E_\kappa R = E_\kappa R E_{\pi(\kappa)}$, while $E_\lambda R E_{\pi(\kappa)} = 0$ for $\lambda \neq \kappa$. Since $R = \sum_\kappa E_\kappa R$, we have $RE_{\pi(\kappa)} = E_\kappa R$.

$$RE_{\pi(\kappa)} = E_\kappa R E_{\pi(\kappa)} = E_\kappa R.$$

The left annihilator $l(A)$ of A vanishes. For, if $l(A) \neq 0$, then its minimal right subideal, contained in R, would be isomorphic to a certain $\bar{e}_\lambda \bar{A}$, which gives a contradiction since $\bar{e}_\lambda \bar{A} A \neq 0$.

$l(A) = l(E)$ as $A = EA$. So $l(E) = 0$ and $A = AE$ (by Peirce decomposition). Thus E is (two-sided) *unit element* in A.

We now construct an algebra A_0 with capacities all equal to 1, which belongs to A in the sense of I, § 3.[7] Namely

$$A_0 = \sum_{\kappa,\lambda} e_{\kappa,1} A e_{\lambda,1}.$$

A_0 satisfies the same assumptions as A. And, in order to secure that A is quasi-Frobeniusean (and Frobeniusean in case of $f(\pi(\kappa))$

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4) Two idempotent elements e and e' in A are called *isomorphic*, if the right ideals eA and $e'A$ are isomorphic, which is equivalent to that the left ideals Ae and Ae' are isomorphic (and which amounts further to that $e\bar{A}$ and $e'\bar{A}$ (or $\bar{A}e$ and $\bar{A}e'$) are isomorphic, with bars — indicating "mod N").

5) The κ capacity is the maximal number of mutually orthogonal idempotent elements isomorphic to $e\kappa$

6) We employ the inequality $\leqq$ (or $\geqq$ alternatively), rather than the equality $=$, in order to insist on exact one half. But this is not essential. And, if we replace $\leqq$ by $=$, then our assumption amounts to assume a permutation for a maximal set of mutually ortnogonal primitive idempotent elements, rather than for that of non-isomorphic ones, which satisfies the similar eondition as above.

7) I, p. 617. Cf. olso C. Nesbitt W. M. Scott, Some remarks on algebtas over an algebraically closed field, Ann. Math. 44 (1943).

$=f(\kappa)$ (for every κ)) it is sufficient to show that A_0 is Frobeniusean. So, we assume in the rest of our proof that $A=A_0$, i.e. that A itself has capacities $f(\kappa)=1$. Now,

$$e_\kappa R = Re_{\pi(\kappa)}$$

as noted before (with E_λ instead of e_λ). Hence $e_r R$ is a two-sided ideal, and its minimal left subideals must be all isomorphic to $\bar{A}\bar{e}_\kappa$. If we put $m_\lambda=(\bar{e}_\lambda\bar{A}:\ F)=(\bar{A}\bar{e}_\lambda:\ F)$, we have

$$m_\kappa \leqq m_{\pi(\kappa)}\,.$$

Since this is the case for every κ, we have necessarily $m_\kappa = m_{\pi(\kappa)}$, and $e_\kappa R$ itself must be minimal (and $\simeq \bar{A}\bar{e}_\kappa$) as left ideal.

Let, on the other hand, $\mathfrak{r}$, $\mathfrak{r}_1$ be two right ideals in A such that $\mathfrak{r}\geqq\mathfrak{r}_1$ and $\mathfrak{r}/\mathfrak{r}_1$ is irreducible. $\mathfrak{r}/\mathfrak{r}_1\simeq e_{\pi(\kappa)}\bar{A}$ with certain κ, and $\mathfrak{r}=\mathfrak{r}_1+sA$ with $s\in\mathfrak{r}e_{\pi(\kappa)}$ but $\notin\mathfrak{r}_1$ (+ being mere module sum, not necessarily direct). So $l(\mathfrak{r})=l(sA)\frown l(\mathfrak{r}_1)=l(s)\frown l(\mathfrak{r}_1)$. The left ideal $l(\mathfrak{r}_1)s$ is isomorphic to $l(\mathfrak{r}_1)/l(\mathfrak{r}_1)\frown l(s)=l(\mathfrak{r}_1)/l(\mathfrak{r})$. On the other hand $l(\mathfrak{r}_1)\,s\subseteq l(\mathfrak{r}_1)\mathfrak{r}$. Here $l(\mathfrak{r}_1)\mathfrak{r}$ is a sum of right ideals isomorphic to $\mathfrak{r}/\mathfrak{r}_1$ $\simeq \bar{e}_{\pi(\kappa)}\bar{A}$, and is so contained in $Re_{\pi(\kappa)}=e_\kappa R$. Thus $l(\mathfrak{r}_1)\,s\subseteq e_\kappa R$, and the left ideal $l(\mathfrak{r}_1)\,s$ is either 0 or $\simeq\bar{A}\bar{e}_\kappa$. The same must be the case for $l(\mathfrak{r}_1)/l(\mathfrak{r})$. In particular $(l(\mathfrak{r}_1)/l(\mathfrak{r}):\ F)=0$ or m_κ, and $\leqq m_\kappa$ $=m_{\pi(\kappa)}=(\mathfrak{r}/\mathfrak{r}_1:\ F)$, in either case.

Consider a composition series $A>\ldots.>\mathfrak{r}>\mathfrak{r}_1>\ldots.>0$ of right ideals of A, and the series $0=l(A)\subseteq\ldots.\subseteq l(\mathfrak{r})\subseteq l(\mathfrak{r}_1)\subseteq\ldots.$ $\subseteq l(0)=A$ of left annihilators. $(l(\mathfrak{r}_1)/l(\mathfrak{r}):\ F)\leqq(\mathfrak{r}/\mathfrak{r}_1:\ F)$. Since this is the case for every composition residue-module $\mathfrak{r}/\mathfrak{r}_1$, we have necessarily, as we see on summing up the inequalities,

$$(l(\mathfrak{r}_1)/l(\mathfrak{r}):\ F)=(\mathfrak{r}/\mathfrak{r}_1:\ F)$$

(and $l(\mathfrak{r}_1)/l(\mathfrak{r})$ is irreducible and $\simeq\bar{A}\bar{e}_\kappa$). We now deduce easily

$$(l(\mathfrak{r}):\ F)+(\mathfrak{r}:\ F)=(A:\ F)$$

(for every right ideal $\mathfrak{r}$ of A).

Applying this last relation to $\mathfrak{r}=L=r(N)$, the right annihilator of N, we have

$$(l(L):\ F)+(L:\ F)=(A:\ F).$$

Here $l(L)=l(r(N))\subseteq N$. Further $R=\sum e_\kappa R$ is left fully reducible too, as was proved above, and therefore $R \subseteq L$. Moreover $R \simeq A/N$ (either as left module or right module) and so $(N: F) + (R: F) = (A: F)$. Hence necessarily $l(L) = N$ and $L = R$. Then each $Ae_{\pi(\kappa)}$ possesses a unique minimal left subideal $Le_{\pi(\kappa)} = Re_{\pi(\kappa)} (= e_\kappa R)$ isomorphic to $\bar{A}\bar{e}_\kappa$. It follows now that A is Frobeniusean, and our theorem is proved.

As an immediate corollary we have

Corollary 1. *A primary algebra A (over a field F) with left unit element is Frobeniusean if (and only if) a right ideal eA with primitive idempotent element e has a unique minimal right subideal.*

More significant, perhaps, is the following

Corollary 2.[8] *An algebra A (over a field F) is Frobeniusean if (and only if) A possesses a left unit element and $R = l(N)$ is a principal right ideal: $l(N) = cA$.*

Proof. If $R = cA$, it is right-homomorphic to A, by $E \to c$ (E being the left unit element); and indeed, to A/N, since $RN = 0$. As $R = \sum e_{\kappa,i}R$ is a direct sum of at least $\sum f(\kappa)$ minimal right ideals, we have necessarily $R \simeq A/N$, and each $e_{\kappa,i}R$ must be irreducible. Since $e_{\kappa,i}R \simeq e_{\kappa,j}R$ and $\bar{e}_{\kappa,i}\bar{A} \not\simeq \bar{e}_{\lambda,j}\bar{A}$ for $\kappa \neq \lambda$, there must exist a permutation π of $(1, 2, \ldots, k)$ such that $e_{\kappa,i}R \simeq \bar{e}_{\pi(\kappa),j}\bar{A}$ and $f(\kappa) = f(\pi(\kappa))$. So A is Frobeniusean by our theorem.

That Theorem 1 and its corollaries 1, 2 do not hold for a general ring with minimum condition may be seen by the following simple example:

Let $\Omega = F(\xi)$ be the field of rational functions in ξ over a field F. $\xi \to \xi^2$ generates an isomorphism ρ of Ω with the subfield $\Omega^\rho = F(\xi^2)$. We have $\Omega = \Omega^\rho \oplus \xi\Omega^\rho$. Let $A = \Omega \oplus u\Omega$ with $\eta u = u\eta^\rho$ ($\eta \in \Omega$), $u^2 = 0$. A is a primary ring with unit element and $N = u\Omega$ forms the radical of A. N is irreducible as right ideal and $\simeq A/N$. Further, $l(N) = N = uA$ is right-principal. But $N = u\Omega^\rho \oplus u\xi\Omega^\rho$ is reducible as left ideal, and A is not quasi-Frobeniusean.

(The same example shows also that the assumption of being an algebra is essential also in the above alluded criterion of quasi-Frobeniusean algebras by annihilation duality for right, say, ideals

8) Cf. G. Azumaya, On almost symmetric algebras, Jap. Journ. Math. 19 (1948), Theorem 1.

only. For, A, N and 0 exhaust right ideals of A, and $r(l(A)) = r(0) = A$, $r(l(N)) = r(N) = N$, $r(l(0)) = r(A) = 0$).

In connection with our corollary 1, it might be without use to give primary algebras in which L and R do not coincide. We give first an example of a one such that $L \subset R$: Let $A = eF \oplus uF \oplus tF \oplus vF \oplus wF$ be an algebra with 5 basic elements e, u, t, v, w over a field F, such that e is a unit element and $u^2 = t$, $uv = w$, $ut = tu = vu = uw = wu = t^2 = tv = vt = tw = wt = v^2 = vw = wv = w^2 = 0$ The associativity is trivial, since a product of three basic elements $\neq e$ always vanishes. We have $N = uF \oplus tF \oplus vF \oplus wF$, $R = tF \oplus vF \oplus wF$ and $L = tF = wF$.

Next we give an example of a one in which $R \not\subseteq L$, $R \not\supseteq L$: Consider the residue-algebra $B = A/tF$. Namely, $B = eF \oplus uF \oplus vF \oplus wF$ and the multiplication is obtained from the above by putting $t = 0$. Then the radical N_B of B is $uF \oplus vF \oplus wF$ and its left and right annihilators R_B, L_B (in B) are $wF \oplus vF$ and $wF \oplus uF$, respectively.

Now,

Theorem 2. *Let A be a Frobeniusean algebra (over a field F). A residue-algebra $A/\mathfrak{z}$, with a two-sided ideal $\mathfrak{z}$, is Frobeniusean, if (and only if) $l(\mathfrak{z})$ is left-principal: $l(\mathfrak{z}) = Aa$.* (This refines I, Theorem 9, and makes up a discord which existed between that theorem and II, Theorem 15.[9])

Proof. Instead of $l(\mathfrak{z}) = Aa$, we assume rather $r(\mathfrak{z}) = bA$, to be in accord with our previous proof to II, Theorem 15.

Let, as in the referred proof, $\{\rho\}$ be the set of those indices ρ such that $E_\rho \not\subseteq \mathfrak{z}$. $\tilde{N} = N + \mathfrak{z}/\mathfrak{z}$ is the radical of $\tilde{A} = A/\mathfrak{z}$, with waves indicating "mod $\mathfrak{z}$". $\tilde{e}_{\rho,i}$ have the same significance for $\tilde{A}$ as $e_{\kappa,i}$ for A. $E_\kappa(r(\mathfrak{z}) \frown M) \neq 0$ if and only if $\kappa \in \{\rho\}$, where $M = r(N) = l(N)$; for, $l(r(\mathfrak{z}) \frown M) = l(r(\mathfrak{z} + N)) = \mathfrak{z} + N$. On the other hand, M is a direct sum of irreducible two-sided ideals $E_\kappa M = ME_{\pi(\kappa)}$. We see easily

$$r(\mathfrak{z}) \frown M = \sum_\rho E_\rho M (= \sum_\rho ME_{\pi(\rho)}),$$

9) In I, Theorem 9 the two-sided principality was assumed, while the left-principality of $l(z)$ and the right-principality of $r(z)$ were assumed (in case of a Frobeniusean *ring*) in II, Theorem 15.

summation being over $\{\rho\}$. (Up to here we have made no use of our assumption $r(\mathfrak{z}) = bA$).

Now, $e_{\kappa,i}b \neq 0$ if and only if $\kappa \in \{\rho\}$. The left ideal $Ae_{\rho,i}b$ is isomorphic to $A/l(e_{\rho,i}b)$ and here $l(e_{\rho,i}b) = l(e_{\rho,i}A) + l(bA) = A(1-e_{\rho,i}) + \mathfrak{z}$; $Ae_{\rho,i}b \simeq \tilde{A}\tilde{e}_{\rho,i}$. Moreover $Ab \simeq A/l(b) = A/\mathfrak{z} = \tilde{A}$. Since Ab is the sum of $Ae_{\rho,i}b$ while $\tilde{A}$ is the direct sum of $\tilde{A}\tilde{e}_{\rho,i}$, we find that Ab is the direct sum of $Ae_{\rho,i}b$. Hence $Ab \cap \mathrm{M}$, the largest fully reducible left subideal of Ab, is a direct sum of at least $\sum_\rho f(\rho)$ minimal left ideals. But $Ab \cap M \subseteq r(\mathfrak{z}) \cap M$ and this latter is the direct sum of exactly $\sum_\rho f(\pi(\rho)) = \sum_\rho f(\rho)$ minimal left ideals $Me_{\pi(\rho),i}$. So $Ab \cap M = r(\mathfrak{z}) \cap M$ and each $Ae_{\rho,i}b$ has only one minimal left subideal. Since $Ae_{\rho,i}b \simeq \tilde{A}\tilde{e}_{\rho,i}$ all the $Ae_{\rho,i}b$ with a fixed ρ are isomorphic. We see easily that there exists a permutation ν of $\{\rho\}$ such that for every ρ, i the unique minimal left subideal $Ae_{\nu(\rho),i}b$ is contained in $ME_{\pi(\rho)}$, whence $\simeq \bar{A}\bar{e}_{\rho,i}$, and that $f(\nu(\rho)) = f(\pi(\rho)) = f(\rho)$. Since $Ae_{\nu(\rho),i}b \simeq \tilde{A}\tilde{e}_{\nu(\rho),i}$ and $\bar{A}\bar{e}_{\rho,i} \simeq \tilde{A}\tilde{e}_{\rho,i}$ (for each of our ρ), this shows that $\tilde{A}$ satisfies the assumptions in our Theorem 1, π replaced by ν and "right" replaced by "left". (These were, in main, reproduction of a part of our previous proof to II, Theorem 15[10]).

$\tilde{A} = A/\mathfrak{z}$ is thus Frobeniusean by our Theorem 1 (in its dual form), and we are through.

10) Its preliminary part and the first half of a), to be axact.