# 67. *A Note on Extensions of Groups.*

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1. If a group $\mathfrak{G}$ contains a normal subgroup $\mathfrak{N}$ and $\mathfrak{G}/\mathfrak{N}$ is isomorphic to $\mathfrak{A}$, we call $\mathfrak{G}$ an ***extension of*** $\mathfrak{N}$ ***by*** $\mathfrak{A}$. The problem of extension is to obtain all extensions of $\mathfrak{N}$ by $\mathfrak{A}$ when $\mathfrak{N}$ and $\mathfrak{A}$ are given. The conditions to determine every extension were at first given by O. Schreier[1] and afterwards by K. Shoda[2] in another way.

This note is devided in two parts. In section 2, we shall show that the problem of extension can be reduced in a sense to the case when $\mathfrak{N}$ is abelian, and in section 3, we shall consider central extensions of $\mathfrak{N}$ by $\mathfrak{A}$ under the assumption that $\mathfrak{N}$ and $\mathfrak{A}$ are both abelian, where an extension of $\mathfrak{N}$ by $\mathfrak{A}$ is called a ***central extension*** when $\mathfrak{N}$ is contained in its center.

2. From the theorem of O. Schreier, any extension of $\mathfrak{N}$ by $\mathfrak{A}$ may be determined by a factor set $\{C_{a,b}\}$ and a homomorphic mapping $\bar{\sigma}$ of $\mathfrak{A}$ into the residue class group of the automorphism group of $\mathfrak{N}$ by its inner automorphism group. We shall call such extension a ***$\bar{\sigma}$-extension.*** Let $\sigma$ be a mapping from $\mathfrak{A}$ into the automorphism group of $\mathfrak{N}$ such that the residue class containing $\sigma\, a)$ is equal to $\bar{\sigma}$ a). Then any $\bar{\sigma}$-extension may be determined by a factor set $\{C_{a,b}\}$ which satisfies the following conditions:

1) $A^{\sigma(a)\sigma(b)} = C_{a,b}^{-1} A^{\sigma(ab)} C_{a,b}$ $(A \in \mathfrak{N};\ a, b \in \mathfrak{A})$

2) $C_{ab,c} C_{a,b}^{\sigma(c)} = C_{a,bc} C_{b,c}$.

We shall call such factor set a *$\sigma$-factor set.*

Theorem 1. *Let $\{C_{a,b}\}$ and $\{D_{a,b}\}$ be two $\sigma$-factor sets. Then the set $\{Z_{a,b} = D_{a,b} C_{a,b}^{-1}\}$ is contained in the center $\mathfrak{Z}$ of $\mathfrak{N}$ and satisfies the following conditions:*

3) $Z_{ab,c} Z_{a,b}^{\sigma(c)} = Z_{a,bc} Z_{b,c}$.

*Conversely, if $\{C_{a,b}\}$ is a $\sigma$-factor set, and if $\{Z_{a,b}\}$ is contained in $\mathfrak{Z}$ and satisfies 3), then $\{D_{a,b} = C_{a,b} Z_{a,b}\}$ is a $\sigma$-factor set.*

Proof. If $\{C_{a,b}\}$ and $\{D_{a,b}\}$ are both $\sigma$-factor sets, then from 1) $C_{a,b}^{-1} A C_{a,b} = D_{a,b}^{-1} A D_{a,b}$ for any $A \in \mathfrak{N}$, hence $Z_{a,b} = D_{a,b} C_{a,b}^{-1} \in \mathfrak{Z}$. Further, since

1) O. Schreier: Über die Erweiterung von Gruppen, Monatshefte für Math. u. Phisik, 34 (1926) 321–346.

2) K. Shoda: Über die Schreiersche Erweiterungstheorie. Proc. Acad. Tokyo (1943) 518–519.

$\{D_{a,b}\}$ satisfies 2), $C_{a,b}C^{\sigma(c)}_{a,b}Z_{ab,c}Z^{\sigma(c)}_{a,b}=C_{ab,c}C_{b,c}Z_{a,bc}Z_{b,c}$ and hence $\{Z_{a,b}\}$ satisfies 3). Conversely, if $\{C_{a,b}\}$ is a $\sigma$-factor set and if $\{Z_{a,b}\}$ is contained in $\mathfrak{Z}$ and satisfies 3), then for $\{D_{a,b}=C_{a,b}Z_{a,b}\}$ the conditions 1) and 2) will be easily verified.

As an immediate consequence of this theorem, we have the

Corollary. *For any $\bar\sigma$-extension $\mathfrak{G}$ of $\mathfrak{N}$ by $\mathfrak{A}$, $\mathfrak{G}/\mathfrak{Z}$ is uniquely determined disregarding isomorphisms.*

If two $\bar\sigma$-extensions $\mathfrak{G}$ and $\mathfrak{G}'$ are mutually isomorphic by a correspondence such that every element of $\mathfrak{N}$ corresponds to itself and the residue class of $\mathfrak{G}$ mod $\mathfrak{N}$ corresponding to $a\mathfrak{G}\mathfrak{A}$ corresponds to such residue class of $\mathfrak{G}'$ mod $\mathfrak{N}$, then we shall say that $\mathfrak{G}$ and $\mathfrak{G}'$ have the same *type*. As is easily verified, two extensions determined by $\sigma$-factor sets $\{C_{a,b}\}$ and $\{D_{a,b}\}$ have the same type if and only if there exists a set $\{Z_a\}$ of elements from $\mathfrak{Z}$ such that $D_{a,b}=C_{a,b}Z^{-1}_{a,b}Z^{\sigma(b)}_{a}Z_b$. In such a case, we say that $\{D_{a,b}\}$ is *associated* to $\{C_{a,b}\}$. This relation satisfies the three conditions of equivalence, and hence we can classify all $\sigma$-factor sets by this relation. The totality of these classes is denoted by $E_\sigma$ ($\mathfrak{N}$, $\mathfrak{A}$), then there exists a one to one correspondence between $E_\sigma$ ($\mathfrak{N}$, $\mathfrak{A}$) and the totality of types of extensions.

Now we shall suppose that there exists at least one $\bar\sigma$-extension of $\mathfrak{N}$ by $\mathfrak{A}$, and select a $\bar\sigma$-factor set $\{C_{a,b}\}$. Then for any $\bar\sigma$-factor set $\{D_{a,b}\}$, $\{Z_{a,b}=D_{a,b}C^{-1}_{a,b}\}$ is a $\bar\sigma$-factor set respecting to $\mathfrak{Z}$, where the homomorphism of $\mathfrak{A}$ in the automorphism group of $\mathfrak{Z}$ induced by $\sigma$ is also denoted by $\sigma$. Further, $\{D_{a,b}\}$ is associated to $\{D'_{a,b}\}$ if and only if $\{Z_{a,b}=D_{a,b}C^{-1}_{a,b}\}$ is associated to $\{Z'_{a,b}=D'_{a,b}C^{-1}_{a,b}\}$. Thus there exists a one to one correspondence between $E_\sigma$ ($\mathfrak{N}$, $\mathfrak{A}$) and $E_\sigma$ ($\mathfrak{Z}$, $\mathfrak{A}$). Accordingly we have;

Theorem 2. *Let $\bar\sigma$ be a homomorphism of $\mathfrak{A}$ in the residue class group of the automorphism group of $\mathfrak{N}$ by its inner automorphism group, and suppose that there exists at least one $\bar\sigma$-extension of $\mathfrak{N}$ by $\mathfrak{A}$. Then there exists a one to one correspondence between types of extensions of $\mathfrak{N}$ by $\mathfrak{A}$ and those of $\mathfrak{Z}$ by $\mathfrak{A}$.*

As is well known, $E_\sigma$ ($\mathfrak{Z}$, $\mathfrak{A}$) forms an abelian group by the definition of products $\{Z_{a,b}\}\times\{Z'_{a,b}\}=\{Z_{a,b}\cdot Z'_{a,b}\}$. This group will be called the *group of $\bar\sigma$-extensions* of $\mathfrak{N}$ by $\mathfrak{A}$ when there exists at least one $\bar\sigma$-extension.

Corollary. *If the center of $\mathfrak{N}$ is unit group, then there exists a unique $\bar\sigma$-extension of $\mathfrak{N}$ by $\mathfrak{A}$ for any $\bar\sigma$.*

Proof. If there exists at least one $\bar\sigma$-extension then the uniqueness is an immediate consequence of theorem 2. We shall prove the existence. By the assumption, the inner automorphism group of $\mathfrak{N}$ is isomorphic to $\mathfrak{N}$. We shall identify this with $\mathfrak{N}$. Let $\mathfrak{C}$ be the kernel of $\bar{\sigma}$ and let $\mathfrak{S}/\mathfrak{N}$ be the image of $\mathfrak{A}$ by $\bar{\sigma}$. Then $\bar{\sigma}$ induces an isomorphim of $\mathfrak{A}/\mathfrak{C}$ on $\mathfrak{S}/\mathfrak{N}$. Let in this isomorphism a residue class $a\,\mathfrak{C}$ of $\mathfrak{A}/\mathfrak{C}$ corresponds to a residue class $\sigma(a)\,\mathfrak{N}$ of $\mathfrak{S}/\mathfrak{N}$. Then the subgroup of $\mathfrak{S}\times\mathfrak{A}$ which consists of all elements with forms $\sigma(a($ $N$ $ac$ ($N\in\mathfrak{N}$, $C\in\mathfrak{C}$) is a $\bar{\sigma}$-extension of $\mathfrak{N}$ by $\mathfrak{A}$.

Corollary. *Let $\mathfrak{A}$ and the center $\mathfrak{Z}$ of $\mathfrak{N}$ have the finite orders $m$ and $n$ respectively, and suppose that $m$ and $n$ are coprime. Then there exists at most one $\bar{\sigma}$-extension of $\mathfrak{N}$ by $\mathfrak{A}$ for any $\bar{\sigma}$.*

Proof. In this case, it will be easily verified that $E_\sigma$ ($\mathfrak{Z}$, $\mathfrak{A}$) is a unit group, and hence our assertion holds.

Specially, if the orders of $\mathfrak{A}$ and $\mathfrak{N}$ are both finite and coprime, then any $\sigma$-extension of $\mathfrak{N}$ by $\mathfrak{A}$, if exists, must be split.[4]

3. In this section, we shall consider central extensions[5] of $\mathfrak{N}$ by $\mathfrak{A}$ under the assumptions that $\mathfrak{N}$ and $\mathfrak{A}$ are both abelian and $\mathfrak{A}$ has a finite number of generators.

First of all, we shall state without proof Shoda's theorem.

Theorem.[6] *Let $\mathfrak{N}$ and $\mathfrak{A}$ be two groups, and $\mathfrak{A}$ be defined by a set of generators $E=\{a_i\}$ and defining relations $R=\{r(a)\}$. Denote by $\mathfrak{F}(E)$ the free group generated by $E$. and by $\mathfrak{R}$ the normal subgroup of $\mathfrak{F}(E)$ generated by $R$ and further by $A(\mathfrak{N})$ the automorphism group of $\mathfrak{N}$. If a homomorphic mapping $a_i\to\alpha_i$ from $\mathfrak{F}(E)$ into $A(\mathfrak{N})$ and a homomorphic mapping $r(a)\to A_r$ from $\mathfrak{R}$ into $\mathfrak{N}$ satisfy the following conditions:*

1) $A a_i r(a) a_i^{-1} = A_{r_\alpha}^{\alpha_i}$

2) $A^{r(\alpha)} = A_r A_r^{-1}$ (*where $A$ is any element of $\mathfrak{N}$ and $r(\alpha)$ is the image of $r(a)$.*)

*then an extension of $\mathfrak{N}$ by $\mathfrak{A}$ may be obtained by introducing the relations: $a_i A a_i^{-1} A^{-\alpha_i}$, $r(a)A_r^{-1}$ in the free product of $\mathfrak{N}$ and $\mathfrak{F}(E)$. Conversely every extension may be obtained in such a way.*

From this theorem, if $\mathfrak{N}$ is abelian, any central extension of $\mathfrak{N}$ by $\mathfrak{A}$ may be determined by a homomorphic mapping from $\mathfrak{R}/\mathfrak{F}(E)\circ\mathfrak{R}$ into $\mathfrak{N}$, where $\mathfrak{F}(E)\circ\mathfrak{R}$ denotes the commutator subgroup of $\mathfrak{F}(E)$ and $\mathfrak{R}$.

3) See footnote 1).
4) See Zassenhaus's "Lehrbuch der Gruppentheorie" p. 125.
5) See section 1.
6) See footnote 2).

Let $\mathfrak{A}=(a_1)\times\ldots\times(a_n)$ be an abelian group and $t_i$ be the order of $a_i$, then $\mathfrak{A}$ may be regarded as defined by a set of generators $E=\{a_1, a_2, \ldots, a_n\}$ and defining relations $r_i=a_i^{t_i}$, $r_{i,k}=a_i a_k a_i^{-1} a_k^{-1}(i, k=1, \ldots, n;\ i<k)$. Let $\mathfrak{F}(E)$ and $\mathfrak{R}$ have the same significances as above, then we have the following lemma.

Lemma. $\mathfrak{R}/\mathfrak{F}(E)_0\mathfrak{R}$ *is isomorphic with* $\mathfrak{W}=(W_1)\times\ldots\times(W_n)\times(W_{1,2})\times(W_{1,3})\times\ldots\times(W_{n-1,n})$, *where* $(W_i)$ $(i=1, 2, \ldots, n)$ *is a cyclic group of order O and* $(W_{i,k})(i, k=1, 2, \ldots, n)$ *is a cyclic group of order* $t_k$.

Proof. We shall denote by $\bar{\mathfrak{R}}$ the residue class group $\mathfrak{R}/\mathfrak{F}(E)_0\mathfrak{R}$. $\bar{\mathfrak{R}}$ is generated by $\bar{r}_i=r_i\,\mathfrak{F}(E)_0\mathfrak{R})$ and $r_{i,k}=r_{i,k}(\mathfrak{F}(E)_0\mathfrak{R})$. The following relations hold in $\bar{\mathfrak{R}}$:

$$r_k^{a_i}=a_i r_k a_i^{-1}=(a_i a_k a_i^{-1})^{t_k}=(r_{i,k}a_k)^{t_k}$$

$$=r_{i,}(a_k r_{i,k} a_k^{-1})(a_k^2 r_{i,k} a_k^{-2})\ldots(a_k^{t_k-1} r_{i,k} a_k^{-(t_k-1)})a_k^{t_k}=r_{i\,k}^{1+a_k+\cdots+a_k^{t_k-1}}r_k$$

Hence, $\bar{r}_k=\bar{r}_{i,k}^{t_k}\bar{r}_k$, that is, $\bar{r}_{i,k}^{t_k}=\bar{e}$ ($\bar{e}$ is the unit element of $\bar{\mathfrak{R}}$). Accordingly by the mapping $W_i\rightarrow\bar{r}_i$, $W_{i,k}\rightarrow\bar{r}_{i,k}$, $\mathfrak{W}$ is homomorphic to $\bar{\mathfrak{R}}$.

Conversely, from the theorem in Zassenhaus's "Lehrbuch der Gruppentheorie" p. 96, we can obtain a central extension of $\mathfrak{W}$ by $\mathfrak{A}$, introducing the relations $a_i W a_i^{-1} W^{-1}$, $a_i a_k a_i^{-1} a_k^{-1} W_{i,k}^{-1}$, $a_i^{t_i} W_i^{-1}$ into the free product of $\mathfrak{F}(E)$ and $\mathfrak{W}$. Hence by the mapping $\bar{r}_i\rightarrow W_i$, $\bar{r}_{i,k}\rightarrow W_{i,k}$, $\bar{\mathfrak{R}}$ is homomorphic with $\mathfrak{W}$. Thus the lemma is proved.

Combining Shoda's theorem and this lemma, we have

Theorem 3. *Let* $\mathfrak{A}$, $\mathfrak{N}$ *and* $\mathfrak{W}$ *have the same significances as above. If a homomorphic mapping from* $\mathfrak{W}$ *into* $\mathfrak{N}$ *is given by the mapping* $W_i\rightarrow A_i$, $W_{i,k}\rightarrow A_{i,k}$, *then, introducing the relations* $a_i^{t_i}A_i^{-1}$, $a_i a_k a_i^{-1} a_k^{-1} A_{i,k}^{-1}$, $a_i A a_i^{-1} A^{-1}$ *in the free product of* $\mathfrak{N}$ *and* $\mathfrak{F}(E)$, *we have a central extension of* $\mathfrak{N}$ *by* $\mathfrak{A}$. *Conversely every central extension of* $\mathfrak{N}$ *by* $\mathfrak{A}$ *may be obtained in such a way.*

By theorem 3, every extension is determined by a set $\{A_i, A_{i,k}\}$ of elements from $\mathfrak{N}$ such that $A_{i,k}^{t_k}=1$ (1 is the unit element of $\mathfrak{N}$). As is easily verified, $\{A_i, A_{i,k}\}$ and $\{B_i, B_{i,k}\}$ determine extensions of the same type if and only if there exist $n$ elements $N_i(i=1, 2, \ldots, n)$ of $\mathfrak{N}$ and the following conditions are satisfied:

1) $B_i = {}_iAN_i^{t_i}$
2) $B_{i,k}=A_{i,k}$.

Hence, we have the following theorem.

Theorem 4. *The group of central extensions* $E_1(\mathfrak{N}, \mathfrak{A})$ *is isomorphic with* $\mathfrak{N}_1/\mathfrak{N}_1^{t_1}\times\ldots\times\mathfrak{N}_n/\mathfrak{N}_n^{t_n}\times\mathfrak{N}_{1,2}\times\ldots\times\mathfrak{N}_{n-1,n}$, *where* $\mathfrak{N}_i(i=1, 2, \ldots, n)$ *is isomorphic with* $\mathfrak{N}$ *and* $\mathfrak{N}_{i,k}(i, k=1, \ldots, n;\ i>k)$ *is isomorphic with the subgroup of* $\mathfrak{N}$ *which consists of all elements whose orders devide* $t_k$.