47. *Brownian Motions in a Lie Group.*

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The notion of Brownian motions has been introduced by N. Wiener [1] [2][1] in the case of the real number space (or more generally the n-space) and by P. Lévy [3] in the case of the circle. We shall here extend this notion in the case of a general Lie group.[2]

§ *1. Definition and fundamental theorems.* Let G be an n-dimensional Lie group. A random process $\pi(t)$ in G is called to be a *right (left) invariant Brownian motion* in G, if it satisfies the following five conditions **M**, **C**, **T**, **S** and **C***.

M. $\pi(t)$ is a simple Markoff process; we denote the transition probability law of $\pi(t)$ with $F(t, p, s. E)$ i.e.

$$F(t, p, s, E) = P_r\{\pi(s)\epsilon E/\pi(t) = p\} .$$

C. Kolmogoroff-Feller's continuity condition [4] [5]. For any neighbourhood U of p it holds that

$$\lim_{s\to t+0} \frac{1}{s-t} F(t, p, s, G-U) = 0$$

and the following limits exist $(1 \leqq i, j \leqq n)$

$$a^i(t, p) \equiv \lim_{s\to t+0} \frac{1}{s-t} \int_U (x^i - x_0^i)\, F(t, x_0, s, dx),$$

$$B^{ij}(t, p) \equiv \lim_{s\to t+0} \frac{1}{s-t} \int_U (x^i - x_0^i)(x^j - x_0^j)\, F(t, x_0, s, dx),$$

where (x^i) is a local coordinate defined on U and (x_0^i) is the coordinate of p.

T. temporal homogeneity. $F(t, p, s, E) = F(t+\tau, p, s+\tau, E)$.

S. spatial homogeneity.

right invariance $F(t, p, s, E) = F(t, pr, s, Er)$.

(left invariance $F(t, p, s, E) = F(t, lp, s, l E)$.)

C* continuity. Almost all sample motions[3] are continuous.

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1) The numbers in [ ] correspond to those in the the references at the end of this paper.

2) Prof. K. Yosida has obtained a similar result in making use of his operator-theoretical method. See the preceding article.

3) In the analytical theory of probability a random motion is represented by a motion depending on a probability parameter. Any motion for each parameter value is called to be a sample motion.

By a *Brownian motion* in G we understand a right invariant one or a left invariant one. A *both-sides invariant Brownian motion* is defined as a Brownian motion which is right invariant as well as left invariant.

Now, put

$$D_t f(p) = \lim_{s\to t} \frac{1}{s-t}\int_G (f(q)-f(p))F(t, p, s, dq)\,. \tag{1.1}$$

Then we see by **C** and **T** that $D_t f(p)$ is written as

$$D_t f(p) = a^i(p)\frac{\partial f}{\partial x^i}(p) + \frac{1}{2}B^{ij}(p)\frac{\partial^2 f}{\partial x^i\,\partial x^j}(p)^{4)} \tag{1.1. a}$$

for any bounded function $f(p)$ of class C_2, where $\|B^{ij}(p)\|$ is a symmetric non-negative-definite matrix by virtue of

$$\xi_i\xi_j B^{ij}(p) = \lim_{s\to t+0}\frac{1}{s-t}\int(\xi_i(x^i-x_0^i))^2 F(t, x_0, s, dx) \geqq 0\,, \tag{1.1. b}$$

namely that D is an elliptic differential operator in G independent of t. Therefore we may eliminate t and write simply as D. D is called to be the *generating operator* of the Brownian motion $\pi(t)$.

We shall here state several fundamental theorems.

Theorem 1. (Characterization of generating operators). Let D be any elliptic differential operator defined for any bounded function of class C_2. Then the following three conditions are equivalent to each other.

(G. 1) D is a generating operator of a right (left) invariant Brownian motion.

(G. 2) D commutes with any right (left) translation operator $R_r(L_l)$, where $R_r f(p) = f(pr)$ $(L_l f(p) = f(lp))$.

(G. 3) D is expressible in the form:

$$D = A^i X_i + \frac{1}{2}B^{ij}X_iX_j \tag{1.2}$$

where $\{X_i\}$ is a basis of the infinitesimal operators of left (right) translations and A^i, B^{ij} are all real constants such that the matrix $\|B^{ij}\|$ is a symmetric non-negative-definite one.

Theorem 2. (A generalization of the Fokker-Planck equation [6]). If we put

$$f(s, p) = \int f(q)F(t, p, s, dq)\ \ (t \leqq s)$$

$f(q)$ being a function of class C_2 which vanishes outsides of a

4) We shall eliminate the summation sign $\sum$ according to the usual rule of tensor caluclus.

compact set, then $f(s, p)$ satisfies the following partial differential equation:

$$(1.3)\qquad \frac{\partial}{\partial s} f(s, p) = Df(s, p)$$

with the initial condition

$$(1.4)\qquad f(t, p) = f(p).$$

Theorem 3. (Uniqueness theorem). The transition probability law of a Brownian motion is uniquely determined by its generating operator.

Theorem 4. (A condition for the both-sides invariance). A necessary and sufficient condition that D be the generating operator of a both-sides invariant Brownian motion is that D is expressible in the form (2) such that $\{A^i\}$ and $\{B^{ij}\}$ satisfies, besides the above-stated conditions,

$$(1.5)\qquad A^j C^l_{kj} = 0, \quad B^{ij} C^l_{kj} + B^{jl} C^i_{kj} = 0 \quad (1 \leq i, k, l \leq n).$$

Theorem 5. (A generalization of "*differential*" property). Let $\pi(t)$ be a right (left) invariant Brownian motion in G. Then

$$\pi(s_i)\,\pi(t_i)^{-1}\,(\pi(t_i)^{-1}\,\pi(s_i)),\ i = 1, 2, \dots, m,$$

are independent G-valued random variables for $t_1 < s_1 \leq t_2 < s_2 \leq \dots \leq t_m < s_m$.

§ 2. *Proof of the theorems.*

Proof of Th. 1. We shall consider only the case of right invariant Brownian motions. It is clear by the definition that (G. 1) implies (G. 2). We shall prove that (G. 2) implies (G. 3)[5]. By (G. 2) we have

$$Df(r) = R_r Df(e) = DR_r f(e) = Df_r(e), \text{ where } f_r(p) \equiv f(pr).$$

By taking an adequate coordinate (x^i) around e we may assume that X_i is expressed as

$$X_i g(x) = c^k_i(x) \frac{\partial g}{\partial x^k}(x),\ c^k_i(e) = \delta^k_i,\ x^i(e) = 0.$$

Then we have

$$Dg(e) = (A^i X_i + \frac{1}{2} B^{ij} X_i X_j) g(e),$$

where

$$A^i = a^i(e) - \frac{1}{2} B^{jk}(e) \frac{\partial c^i_k}{\partial x^j}(e),\ B^{ij} = B^{ij}(e).$$

5) The author's original proof was the same in essential as that stated here but more complicated. He owes much to M. Kuranishi for the simplification of the proof.

$\| B^{ij} \|$ is clearly a symmetric non-negative-definite matrix by the definition.

Therefore $Df_r(e)$ is written as

$$Df_r(e) = (A^i X_i + \frac{1}{2} B^{ij} X_i X_j) f_r(e) ,$$

where A^i, B^{ij} satisfy the conditions stated in (G. 3). In considering that X_i is an infinitesimal operators of left translations and so commutative with R_r, we obtain

$$Df(r) = (A^i X_i + \frac{1}{2} B^{ij} X_i X_j) f_r(e) = (A^i X_i + \frac{1}{2} B^{ij} X_i X_j) f(r) .$$

Next, we shall prove that (G. 3) implies (G. 1). K. Yosida has shown, in making use of his operator theoretical method, that (G. 3) implies that D is the generating operator of a simple Markoff process which satisfies **M**, **C**, **T** and **S**. By the use of a stochastic differential equation [7] we shall here show that D is the generating operator of a right invariant Brownian motion, which satisfies **C*** besides the above four conditions; this will mean that (G. 3) implies (G. 1). We fix a *canonical coordinate* [7] (x^i) around e, and define a canonical coordinate around p by

$$x_p^i(qp) = x^i(q), \qquad 1 \leqq i \leqq n. \tag{2.1}$$

Then $\{(x_p^i),\ p \in G\}$ is a *canonical coordinate system* [7]. By the above argument we see by (G. 2) that

$$Df(p) = \left(a^i \frac{\partial}{\partial x_p^i} + \frac{1}{2} B^{ij} \frac{\partial^2}{\partial x_p^i \partial x_p^j}\right) f(p) , \tag{2.2}$$

where a^i, B^{ij} are all independent of p and $\| B^{ij} \|$ is a nonnegative definite matrix. We fix a real matrix $\| b_j^i \|$, such that

$$b_k^i b_k^j = B^{ij} . \tag{2.3}$$

Now we shall consider an arbitrary local coordinate (x^i) whose coordinate neighbourhood contains p. For this coordinate we define

$$a^i(x) = a^j \frac{\partial x^i}{\partial x_p^j} + \frac{1}{2} b_k^j b_k^l \frac{\partial^2 x^i}{\partial x_p^j \partial x_p^l}, \quad b_k^i(x) = b_k^j \frac{\partial x^i}{\partial x_p^j} . \tag{2.4}$$

Then we have

$$\begin{aligned} & a^i(p) \frac{\partial f}{\partial x^i}(p) + \frac{1}{2} b_k^i(p) b_k^j(p) \frac{\partial^2 f}{\partial x^i \partial x^j}(p) \\ & \quad = a^i(p) \frac{\partial f}{\partial x_p^i}(p) + \frac{1}{2} b_k^i(p) b_k^j(p) \frac{\partial^2 f}{\partial x_p^i \partial x_p^j}(p) = Df(p) . \end{aligned}$$

Since $Df(p) \equiv (A^i X_i + \frac{1}{2} B^{ij} X_i X_j) f(p)$ is independent of the special choice of the local coordinate, it is so with the left side of the above equation. This implies that $a^i(x)$ is transformed in the following manner:

$$\text{(2.5. a)} \qquad \bar{a}^{i}(x) = a^j(x)\frac{\partial \bar{x}^i}{\partial x^j} + \frac{1}{2} b^j_l(x) b^k_l(x) \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} .$$

$b^i_j(x)$ is clearly transformed as follows by (2.4):

$$\text{(2.5. b)} \qquad \bar{b}^i_k(x) = b^j_k(x) \frac{\partial \bar{x}^i}{\partial x^j} .$$

Thus we may consider the following *stochastic differential equation* [7]:

$$\text{(2.6)} \qquad d\xi^i(t) = a^i_j(\xi(t))\, dt + b^i_j(\xi(t))\, d\beta^i(t) ,$$

$(\xi^i(t))$ being a local coordinate of a random motion $\pi(t)$ in G.

In order to show the existence of the solution of this equation we shall verify the conditions (3.8), (3.9) and (3.10) in Theorem 3.1 in [7]. (3.9) and (3.10) are evident. We shall easily verify (3.8) in considering that $a^i(x)$, $b^i_j(x)$ is determined by the same expression around every point with respect to the above canonical coordinate system by virtue of the definitions.

By Theorem 3.2 in [7] we see that the solution $\pi(t)$ is a continuous simple Markoff process whose transition probability law $F(t, p, s, E)$ satifies

$$\text{(2.7)} \qquad \lim_{s\to t+0} \frac{1}{s-t}\int (f(q)-f(p))\, F(t, p, s, dq)$$
$$= a^i(p)\frac{\partial f}{\partial x^i}(p) + \frac{1}{2} b^i_k(p) b^j_k(p) \frac{\partial^2 f}{\partial x^i \partial x^j}(p) = Df(p).$$

Thus we see that $\pi(t)$ satisfies the conditions **M, C, C*** in § 1. By comparing the solution of (2.6) with the initial condition:

$$\text{(2.8. a)} \qquad \pi(t) = p$$

with the solution of the same equation with the initial condition:

$$\text{(2.8. b)} \qquad \pi(t+\sigma) = p$$

and in remembering the temporal homogeneity of $(\beta^i(t))$, we can easily verify that $\pi(t)$ is temporally homogeneous. In order to show the spatial homogeneity we need only to remember that, if $\pi(\tau)$ is the solution of (2.6) with the initial condition: $\pi(t) = p$, then $\pi^*(\tau) \equiv \pi(\tau)r$ is the solution of (2.6) with $\pi^*(t) = pr$.

Proof of Th. 2. By the right-invariance we see that

$$f(s, p) = \int f(q\cdot p) F(t, p, s, dq\cdot p) = \int f(q\cdot p) F(t, e, s, dq) ,$$

which implies that $f(s, p)$ is a bounded function of class C_2 in p.

By the temporal homogeneity we have

$$\begin{aligned} f(s+\Delta, p) &= \int f(q)F(t, p, s+\Delta, dq) \\ &= \iint f(q)F(t, p, t+\Delta, dr)F(t+\Delta, r, s+\Delta, dq) \\ &= \iint f(q)F(t, p, t+\Delta, dr)F(t, r, s, dq) \\ &= \int f(s, r)F(t, p, t+\Delta, dr) \end{aligned}$$

and so

$$f(s+\Delta, p)-f(s, p) = \int (f(s, r)-f(s, p))F(t, p, t+\Delta, dr)$$

and accordingly

$$\lim_{\Delta\to+0} \frac{f(s+\Delta, p)-f(s, p)}{\Delta} = Df(s, p)$$

$Df(s, p)$ being continuous in s as is easily verified, we obtain (1.3) from the above identity.

Proof of Th. 3. Let $F_1(t, p, s, E)$ and $F_2(t, p, s, E)$ be the transition probability law of the Brownian motions with the same generating operator D. We shall here prove that $F_1 = F_2$. For this it is sufficient to show that the functions

$$f_i(s,p) = \int f(q)F_i(t, p, s, dq),\ i = 1, 2,$$

coincide with one another for any function f of class C_2 which vanishes outsides of a compact set. Put

(2.9) $$g(s, p) = e^{-s}(f_1(s, p)-f_2(s,p)).$$

Then $g(s, p)$ is the solution of the equation

(2.10) $$\frac{\partial}{\partial s}g(s, p) = -g(s, p) + Dg(s, p)$$

with the initial condition:

(2.11) $$g(t, p) = 0.$$

Since $|f_1(s, p) - f_2(s, p)| \leqq 2\max|f(p)|$, $g(s, p)$ tends to 0 uniformly in p as $s\to\infty$. When G is compact, $g(s, p)$ takes the maximum in $s \geqq t$, $p \in G$. When G is not compact (but locally compact as a Lie group), we also see that $g(s, p)$ takes the maximum, in considering that

$$g(s, p) \equiv e^{-s}\left(\int f(q\cdot p)F_1(t, e, s, dq)-\int f(q\cdot p)F_2(t, e, s, dq)\right)$$

tends to 0 uniformly in $t \leqq s \leqq t'$ (t' being any assigned constant) as p tends to the point at infinity of G. Let $g(s_0, p_0)$ be the maximum. When $s_0 = t$, we have $g(s_0, p_0) = 0$ by (2.11). When $s_0 > t$, we have

$$\frac{\partial}{\partial s} g(s_0, p_0) = 0 \ , \ Dg(s_0, p_0) \leqq 0$$

in remembering the expression (2.2) of D. Therefore we see, by virtue of (2.10), that $g(s_0, p_0) \leqq 0$. Thus we see that $g(s, p) \leqq g(s_0, p_0) \leqq 0$. Similarly we obtain $g(s, p) \geqq 0$ in considering the minimum of $g(s, p)$. Consequently we have $g(s, p) \equiv 0$, i.e. $f_1(s, p) \equiv f_2(s, p)$.

Proof of Th. 4. Let D be the generating operator of a both-sides invariant Brownian motion. Then we see, by Theorem 1, that D is expressible in the form (1.2) and commutative with each X_i. Therefore we have (C^i_{jk} = structural constants)

$$\begin{aligned} 0 &= DX_k - X_k D \\ &= A^i[X_i, X_k] + \frac{1}{2} B^{ij} X_i [X_j, X_k] + \frac{1}{2} B^{ij} [X_i, X_k] X_j \\ &= A^j C^l_{kj} X_l + \frac{1}{2} (B^{ij} C^l_{kj} + B^{jl} C^i_{kj}) X_i X_l \end{aligned}$$

and so we obtain (1.5). Thus the *necessity* is proved. By the above argument, the *sufficiency* is also evident.

We may easily show *Th. 5* by making use of the spatial homogeineity of $\pi(t)$.

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