18. On a Theorem Concerning the Homological Structure and the Holonomy Groups of Closed Orientable Symmetric Spaces.

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1. In his interesting manuscript "On the relation between homological structure of Riemannian spaces and exact differential forms which are invariant under holonomy groups" [6][1] written in Japanese, the late Mr. Iwamoto proved the following theorem: "Let B_p be the pth Betti number of a closed orientable Riemannian manifold M_n and B'_p the maximum number of linearly independent (in the sense of algebra) differential forms of rank p which are invariant under the holonomy group h of M_n, then $B_p \geqq B'_p$". As the skew symmetric tensors which are coefficients of differential forms Π invariant under the holonomy group h are covariant constant, Π's are harmonic differential forms. The above theorem is an immediate consequence of Hodge's theorem [5], for two distinct harmonic differential forms of rank p cannot be homologus.

In connexion with the above theorem, he stated without any indication of the proof the following:

Theorem: *If the Riemannian manifold in consideration is symmetric in the sense of Cartan, then,* $B_p = B'_p$.

The purpose of this paper is to give the proof of this theorem.

2. We shall start with the group theoretical definition of symmetric Riemannian spaces.

Let M_n be an n-dimensional homogeneous space with the Lie group of structure G and O be a point of M_n. Then all transformations of G which leave O unaltered constitute the group of isotropy g of M_n. Now, a one to one mapping π of G (as a topological space) onto itself which satisfies the properties (i) $\pi^2 = 1$ (involutive property), (ii) conservation of the law of composition, is called an involutive automorphism of G. It is evident that all elements of G which are invariant under π constitute a group, we shall call it the characteristic subgroup of G with respect to π. If the characteristic subgroup of G with respect to π coincides with the group of isotropy g, then we call M_n a symmetric space.

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1) The brackets [ ] denote the order of papers arranged in the bibliography at the end of this paper.

If we choose suitable bases for generating operators of G so that G is generated by $X_1, \ldots, X_n, \ldots, X_r$ and g by $X_{n+1}, \ldots, X_r$, then π causes the following linear transformation on the canonical parameters of G [3] [4]:

$$e'^i = -e^i, \quad e'^\alpha = e^\alpha \;^{2)}$$

Moreover, if g is compact, we can introduce in M_n, by virtue of Weyl's theorem [4] on compact group of linear transformations, a Riemannian metric which is invariant under G. The symmetric space M_n endowed with such Riemannian metric is the symmetric Riemannian space in consideration.

The set of linear transformations for e^i induced by transformations of the linear adjoint group of G corresponding to inner automorphisms of G induced by elements of g is called the linear group of isotropy γ of M_n. According to Cartan's theory [3], invariant differential forms of M_n under G are representable as exterior forms of n components ω^i of infinitesimal transformations of G with constant coefficients such that they are invariant under all linear transformations for ω^i of the linear group of isotropy γ.

Moreover, if we call homogeneous spaces with compact group of structure as isogeneous spaces, then the p th Betti number B_p of any compact and isogeneous symmetric space is equal to the maximum number of linearly independent (in the sense of algebra) invariant differential forms of M_n under G. Accordingly, the p th Betti number B_p of any compact and isogeneous symmetric space is given as the maximum number of linearly independent exterior forms of e^i of rank p with constant coefficients such that they are invariant under all linear transformations for e^i of the linear group of isotropy. γ.

3. On the other hand, let Π be a differential form of rank p which is invariant under the holonomy group h of the symmetric Riemannian space in consideration. Then, the coefficients of Π constitute a covariant constant skew-symmetric tensor, hence Π is exact. Now let us consider Π at the point O of M_n and denote it by Π°. If we take the generators $X_1, \ldots, X_n$ at O as a frame of reference, then Π° can be written as follows:

$$\Pi^\circ = a_{i_1, \ldots, i_p} \omega^{i_1} \ldots \omega^{i_p}.$$

If we perform transformations of the holonomy group h for ω^i then Π° is invariant under h. Let C be a curve which joins O

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2) We assume that the indices $\begin{cases} i, \; j, \; k \\ \alpha, \; \beta, \; r \\ A, \; B, \; C \end{cases}$ take values $\begin{cases} 1, \; 2, \ldots, n \\ n+1, \ldots, r \\ 1, \; 2, \ldots, r \end{cases}$ respectively.

to an arbitrary point P of M_n. We transport frames, vectors and tensors at O parallel to P, then we get a differential form $\Pi(C, P)$ at P having the same values as those of Π° for corresponding infinitesimal vectors at P and O. It may be seen that $\Pi(C, P)$ at P depends on the curve C joining the point O to P. But we can observe that $\Pi(C, P)$ does not depend on C by virtue of the fact that Π° is invariant under the holonomy group h. The differential form $\Pi(P)$ thus defined is the differential form Π in consideration.

Hence, the problem of finding differential forms of rank p in M_n which are invariant under the holonomy group reduces to the purely algebraic problem of finding exterior forms of rank p with costant coefficients such that they are invariant, when the n variables e^i undergo linear transformations of h. Consequently, if we success to prove the coincidence of the group of isotropy γ and the holonomy group h for any closed orientable and symmetric Riemannian space, it is evident that our assertion is true.

4. We shall now prove that "for any closed orientable and symmetric Riemannian space M_n, the group of isotropy coincides with the holonomy group h".

Let us take bases for the group G as was stated in no. 2. Then it is easily seen that for the constants of structure $C_{ABC}(=-C_{BAC})$ the following relation holds good:

$$C_{ijk} = C_{i\alpha\beta} = C_{\alpha\beta k} = 0.$$

As the generating operators of the linear adjoint group are

$$E_B = \sum e^A C_{ABC} \frac{\partial}{\partial e^C},$$

the infinitesimal transformations of the linear group of isotropy γ are given by

$$E_\alpha = \sum e^j C_{j\alpha i} \frac{\partial}{\partial e^i}.$$

Now, as M_n is closed and g is compact by hypothesis, G is also compact. Hence the linear adjoint group is compact. Accordingly, by Weyl's theorem there exists a positive definite quadratic form of e^A which is invariant under the linear adjoint group. We can choose without any loss of generality bases for G so that the invariant quadratic form take the form $e_1^2+\cdots+e_r^2$. Then $e_1^2+\cdots+e_n^2$ is also invariant under the group of isotropy γ. If we choose such bases for G, then we can easily verify that the following relations hold good:

$$(1)\qquad \begin{aligned} &C_{\alpha ij}+C_{\alpha ji}=0,\ \ C_{\alpha\beta\gamma}+C_{\alpha\gamma\beta}=0,\\ &C_{i\alpha k}+C_{ik\alpha}=0. \end{aligned}$$

The equations of Maurer-Cartan of G are given by

$$(2)\qquad \begin{cases} (\omega^k)'=\sum C_{i\alpha k}[\omega^i\omega^\alpha],\\ (\omega^\gamma)'=\sum\limits_{(ij)} C_{ij\gamma}[\omega^i\omega^j]+\sum\limits_{(\alpha\beta)} C_{\alpha\beta\gamma}[\omega^\alpha\omega^\beta]. \end{cases}$$

where dashes mean exterior derivations.

The metric of the Riemannian space M_n in consideration is given by

$$ds^2=\sum_{i=1}^{n}(\omega^i)^2.$$

The condition that M_n has no torsion is given by

$$(\omega^k)'=[\omega^i\omega_{ik}].$$

Comparing the last equation with (2), we get

$$(3)\qquad \omega_{ik}=C_{i\alpha k}\omega_\alpha\,.$$

The curvature of M_n is also given by

$$\Omega_{ij}=-\omega'_{ij}+[\omega_{ik}\omega_{kj}].$$

If we put (3) into the last equation we get

$$\begin{aligned}\Omega_{ij}=C_{\gamma ij}\Big(&\sum_{(kh)}C_{kh\gamma}[\omega^k\omega^h]+\sum_{(\alpha\beta)}C_{\alpha\beta\gamma}[\omega^\alpha\omega^\beta]\Big)\\ &+\sum_{\alpha,\beta}C_{\alpha ik}C_{\beta kj}[\omega^\alpha\omega^\beta].\end{aligned}$$

As the coefficient of $[\omega^\alpha\omega^\beta]$ vanishes, on account of the Jacobi's identity, we get (cf. [2])

$$\Omega_{ij}=\sum_\gamma C_{\gamma ij}C_{\gamma kh}[\omega^k\omega^h].$$

Consequently, the infinitesimal transformation of the holonomy group h corresponding to an elementary cycle in M_n is given by

$$(4)\qquad \Delta e^i=C_{\gamma kh}C_{\gamma ij}e^j.$$

The right hand member of the last equation can be written as $C_{\gamma kh}E_\gamma(e^i)$. We shall write

$$H_{kh}=C_{\gamma kh}E_\gamma\,.$$

Then we see that

$$\begin{aligned}(H_{kh}\,,\,E_\beta)&=C_{\alpha kh}C_{\alpha\beta\gamma}E_\gamma\\ &=C_{\beta ih}H_{ki}-C_{\beta ki}H_{ih}\,,\end{aligned}$$

hence the set of generators $C_{\gamma kh}E_\gamma$ constitutes an invariant subgroup of the linear group of isotropy.

In virtue of (3), the Riemannian space in consideration can be regarded as a non-holonomic space having the linear group of isotropy as its fundamental group. Consequently the invariant subgroup of γ in consideration is nothing but the holonomy group h [1].

On the other hand, the linear group of isotropy γ depends on $(r-n)$ essential parameters as well as g. Hence the $(r-n)$ operators E_α are linearly independent, so that the matrix $\| C_{\gamma(kh)} \|$ has maximum rank. Accordingly, we can express E_α linearly in terms of H_{kh}. Consequently, the linear group of isotropy γ coincides with the holonomy group h.

References.

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