15. *On the Behaviour of Power Series on the Boundary of the Sphere of Analyticity in Abstract Spaces.*

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In classical analysis there exists a singular point at least on the circle of convergence of the power series, but this is not true generally in the case of the power series in complex Banach spaces. In this paper we shall investigate a necessary and sufficient condition for power series in complex Banach spaces to be analytic at all points on the boundary of the sphere of analyticity.

Let E and E' be two complex Banach spaces and an E'-valued function $h_n(x)$ defined on E be a homogeneous polynomial of degree n. Then the radius of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$ exists, which is written by τ*). The sphere $\|x\| < \tau$ is called the sphere of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$.

Theorem 1. In order that $\sum_{n=0}^{\infty} h_n(x)$ is analytic at all points on the boundary of the sphere of analyticity, it is necessary and sufficient that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} < \frac{1}{\tau}, \quad \dots\dots\dots\dots\dots \quad (1)$$

for an arbitrary compact set G extracted from the set $\|x\| = 1$.

Proof. Let $\sum_{n=0}^{\infty} h_n(x)$ be analytic at all points on $\|x\| = \tau$. If a compact set G extracted from $\|x\| = 1$ exists and it satisfies the following equality

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} = \frac{1}{\tau},$$

we have

$$\frac{1}{\tau+\varepsilon_i} < \sqrt[n_i]{\sup_{x\in G} \|h_{n_i}(x)\|} \quad \dots\dots\dots\dots\dots \quad (2)$$

for a sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_n > \dots$ which tends to zero and for n_i which corresponds to ε_i, where $i = 1, 2, \dots$

Since G is compact, there exists x_i in G which satisfies $\sup_{x\in G} \|h_{n_i}(x)\| = \|h_{n_i}(x_i)\|$. Then we can select from $\{x_i\}$ a subsequence which converges to x_0 and, of course, $x_0 \in G$. In order not to change notation we shall suppose simply that the sequence $\{x_i\}$ itself converges to x_0. Put $y_i = (\tau+\varepsilon_i)x_i$ and $y_0 = \tau x_0$, then y_i converges to y_0. From (2), we have

$$1 < \| h_{n_i}(y_i) \| \quad \dots\dots\dots\dots\dots\dots \quad (3)$$

where $i = 1, 2, \dots$

Let M be a compact set composed of $y_0\, e^{i\theta}$, where $0 \leqslant \theta \leqslant 2\pi$. Since $\sum_{n=0}^{\infty} h_n(x)$ is analytic on M, we can find a finite system of neighborhoods U_i of $y_0 e^{i\theta_i} (i = 1, 2, \dots, n_0)$ such that $\sum_{i=1}^{n_0} U_i$ covers M and $\| \sum_{n=0}^{\infty} h_n(y) \| \leqslant N$ for $y \in \sum_{i=1}^{n_0} U_i$. Now we choose two positive numbers δ and ρ, such that $y\alpha \in \sum_{i=1}^{n_0} U_i$, when $\| y-y_0 \| \leqslant \rho$ and $\| \alpha \| = 1+\delta$. Then we have

$$\| h_n(y) \| = \| \frac{1}{2\pi i} \int_{|\alpha|=1+\delta} \frac{\sum_{n=0}^{\infty} h_n(y\alpha)}{\alpha^{n+1}} d\alpha \| \leqslant \frac{N}{(1+\delta)^n}, \quad \dots \quad (4)$$

for $n = 1, 2, \dots$ and $\| y-y_0 \| < \rho$. Since y_i converges to y_0, (4) contradicts to (3). This shows that the condition (1) is necessary.

Let y_0 be an arbitrary point on $\| y \| = \tau$. Suppose that there exists a sequence $\{y_n\}$ which converges to y_0 and satisfies the following inequalities

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(y_i) \|} \geqslant 1-\varepsilon_i, \quad \dots\dots\dots\dots\dots \quad (5)$$

for $i = 1, 2, \dots$, where a sequence of positive numbers $\{\varepsilon_i\}$ converges to zero with $\varepsilon_{i+1} < \varepsilon_i$. Put $\frac{y_i}{\| y_i \|} = x_i$ and $\{x_i\} = G$. Then G is a compact set on $\| x \| = 1$. Now we assume (1). Then there exists a positive number ε, such that $\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \| h_n(x) \|} \leqslant \frac{1}{\tau+3\varepsilon}$. From this, we have $\| h_n(x_i) \| \leqslant \frac{1}{(\tau+2\varepsilon)^n}$, for $n \geqslant n_0$ and $i = 1, 2, \dots$

On the other hand, there exists N such that $\| y_i \| < \tau+\varepsilon$ for $i \geqslant N$, because $y_i \to y_0$ and $\| y_0 \| = \tau$. Thus we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(y_i) \|} \leqslant \frac{\tau+\varepsilon}{\tau+2\varepsilon},$$

for $i \geqslant N$, contradicting to (5).

From this we can easily see that there exist two positive numbers δ and ε, such that $\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(y) \|} \leqslant 1-\varepsilon$ uniformly for

$\|y-y_0\|<\delta$. Hence, $\sum_{n=0}^{\infty} h_n(x)$ is uniformly convergent in $\|y-y_0\|<\delta$ and then $\sum_{n=0}^{\infty} h_n(x)$ is analytic in $\|y-y_0\|<\delta$. This completes the proof, since y_0 is an arbitrary point on $\|x\|=\tau$.

An example is afforded which is analytic at all points on the boundary of the sphere of analyticity. Put $h_n(x)=\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n$ where $x=(x_1, x_2, \dots)$ is a point of complex-l_2-spaces, and $h_n(x)$ takes complex numbers as its values. Then the radius of analyticity of $\sum_{n=2}^{\infty} h_n(x)$ is 1 and yet $\sum_{n=2}^{\infty} h_n(x)$ is analytic everywhere on $\|x\|=1$. The radius of analyticity of $\sum_{n=2}^{\infty} h_n(x)$ is given by

$$\frac{1}{\tau}=\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\,{}^{*)}.$$

Since $\|x\|=1$, $|x_i|\leqslant 1$ for $i=1, 2, \dots$ Therefore

$$\frac{1}{\tau}=\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\left|\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n\right|}$$

$$\leqslant \lim_{n\to\infty}\sqrt[n]{n\left(1-\frac{1}{n}\right)^n}$$

$$=1 \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (6)$$

Now put $X_m=(0,\dots 0, 1, 0, \dots)$, where only the m-th coordinate is 1 and the others are all zero. Since $\|X_m\|=1$, we have

$$\frac{1}{\tau}\geqslant\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(X_m)\|}=1-\frac{1}{m},$$

for $m=2, 3, \dots$. Hence, we see that $\tau=1$ from (6). Let G be an arbitrary compact set on $\|x\|=1$, then there exists the convergent series of non-negative constants $\sum_{n=1}^{\infty} a_n^2=1$ such that $\sum_{n=m}^{\infty}|x_n|^2\leqslant\sum_{n=m}^{\infty}a_n^2$ for $x\in G$ and $m=1, 2, \dots$. If $a_1=a_2=a_3=\dots=a_{n_0-1}=0$ and $a_{n_0}\neq 0$, $|x_m|^2\leqslant 1$ for $m=1, 2, \dots, n_0$, and $|x_m|^2\leqslant\sum_{n=m}^{\infty}|x_n|^2\leqslant\sum_{n=n_0+1}^{\infty}a_n^2$ for $m\geqslant n_0+1$. Put $\delta=\max\left(1-\frac{1}{n_0}, \sqrt{\sum_{n=n_0+1}^{\infty}a_n^2}\right)$, then $\delta<1$. Thus we have

$$\|h_n(x)\|=\left|\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_n^m\right|<n\delta^n.$$

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*) Isae Shimoda: On power series in abstract spaces, Mathematica Japonicae Vol. 1, No. 2.

Hence, $$\overline{\lim_{n\to\infty}}\sqrt[n]{\sup_{x\in G}\|h_n(x)\|}\leqq\delta<1.$$

Thus Theorem 1 is applicable, and we see that $\sum_{n=2}^{\infty}h_n(x)$ is analytic everywhere on the boundary of the sphere of analyticity. From Theorem 1, we have easily following theorems.

Theorem 2. If a compact set G exists on $\|x\|=1$ which satisfies the following equality $\overline{\lim_{n\to\infty}}\sqrt[n]{\sup_{x\in G}\|h_n(x)\|}=\frac{1}{\tau}$, then there exists at least a singular point on $\|x\|=\tau$. The inverse is also true.

Theorem 3. If a power series $\sum_{n=0}^{\infty}h_n(x)$ is analytic at all points on the boundary of its sphere of analyticity $\|x\|=\tau$, then we have

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}<\frac{1}{\tau}\quad\text{.,..................}\quad(7)$$

for an arbitray point x on $\|x\|=1$.

Theorem 4. If a point x, which lies on $\|x\|=1$, satisfies the following equality $\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}=\frac{1}{\tau}$, then there exists at least a singular point on $\|x\|=\tau$.

The condition (7) is necessary for $\sum_{n=0}^{\infty}h_n(x)$ to be analytic on the boundary of its sphere of analyticity, but is not sufficient as the following example shows.

Put $h_n(X)=x^{n-1}y$ in the complex-2-dimensional spaces, then $h_n(X)$ is a homogeneous polynomial of degree n, where $X=(x, y)$. Then we can easily see that the radius of analyticity of $\sum_{n=1}^{\infty}h_n(X)$ is 1. Let G be a compact set on $\|X\|=1$ composed of $X_0=(1, 0)$ and $X_m=\left(\sqrt{1-\frac{1}{m}}, \sqrt{\frac{1}{m}}\right)$ with $m=1, 2,\dots$ Then we have

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\sup_{x\in G}\|h_n(X)\|}=\lim_{n\to\infty}\sqrt[2n]{\left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n}}=1.$$

On the other hand, we have

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(X)\|}<1,$$

for an arbitrary point X on $\|X\|=1$.

This shows that $\sum_{n=1}^{\infty}h_n(X)$ satisfies the condition (7) and yet has a singular point at least on the boundary of its sphere of analyticity.