13. On a Locally Compact Group with a Neighbourhood Invariant under the Inner-automorphisms.

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Prof. H. Freudenthal[1] proved that a locally compact connected group with an arbitrarily small neighbourhood invariant under the inner-automorphisms is isomorphic to the direct product of a vector group and a compact group.

As an extention of the above theorem the author will prove the following:

Theorem. *Let G be a locally compact connected group with a fixed neighbourhood U. If U is invariant under the inner-automorphisms, then G contains a compact normal subgroup N such that G/N is isomorphic to the direct product of a vector group and a compact group.*

Proof. Let m be the left-invariant Haar measure with a real-valued function $\Delta(x)$ on G such that for an open set V,

$$m(Vx) = m(V)\Delta(x).$$

Then $\Delta(x) = 1$ because

$$m(U)\Delta(x) = m(Ux) = m(xU) = m(U).$$

We see therefore that m is at the same time right-invariant.

Without loss of generality we may assume that U is regularly open.

Put

$$N = \{x\,;\ m(xU \cup U - xU \cap U) = 0\}.$$

Then N coincides with the set $\{x\,;\ xU = U\}$ since U is a regularly open set. Clearly N is a closed subgroup. Furthermore N is compact and normal since $N \subset UU^{-1}$ and

$$a^{-1}xaU = a^{-1}xUa = a^{-1}Ua = U.$$

Let us introduce a metric $d(X, Y)$[2] into the factor group G/N by

$$d(X, Y) = m(xU \cup yU - xU \cap yU).$$

Clearly this metric is left-invariant. Moreover this is right-invariant, for

$$\begin{aligned} d(XA,\ YA) &= m(xaU\cup yaU - xaU\cap yaU) \\ &= m(xUa\cup yUa - xUa\cap yUa) \\ &= m(xU\cup yU - xU\cap yU) \\ &= d(X,\ Y). \end{aligned}$$

Hence by the Freudenthal's theorem[1] G/N is isomorphic to the direct product of a vector group and a compact group, which completes the proof.

Corollary. *Under the assumption of the theorem the closure of the commutator subgroup of G is compact.*[3]

Added in proof. Dr K. Iwasawa kindly informed to the author that he had proved the same theorem independently.

References.

1) H. Freudenthal: *Topologischen Gruppen mit genügend vielen fastperiodischen Funktionen,* Annals of Math. Vol. 37 (1936) 57-75.

2) The capital letters denote the cosets containing the elements represented by the small letters.

3) The author was told that this theorem was proved by Dr. Mostow for the Lie group case.