35. *On Integral Representations of Bilinear Functionals.*

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Let E_1 and E_2 be Banach spaces. Let $A = A(f, g)$ ($f \in E_1$, $g \in E_2$) be a functional defined in (E_1, E_2) such that

1. $A(f, g)$ is additive and homogeneous concerning f and g,
2. $A(f, g)$ is bounded; that is, there is a constant M such that

$$|A(f, g)| \leqq M\|f\|\cdot\|g\| .$$

for all $f \in E_1$, and all $g \in E_2$.

Such functional $A(f, g)$ of f and g is called *bilinear* in (E_1, E_2), and the greatest lower bound of M is called the *norm* of the functional A and is denoted by $\|A\|$. It is then evident that

$$\|A\| = \sup_{\|f\|=1,\, \|g\|=1} |A(f, g)| .$$

Integral representations of bilinear functionals have been studied by Prof. Izumi [1], but their norms were not exactly given. In this note we will give the exact form of their norms. For this purpose we first prove that the representation problem of bilinear functionals in (E_1, E_2) is equivalent to that of the linear operations from E_1 to $\bar{E}_2$ (or from E_2 to $\bar{E}_1$), where $\bar{E}$ denotes the conjugate space of E. Therefore we get the representations of linear operations between some concrete Banach spaces from results of Prof. Izumi [1]. On the other hand, we obtain the general form of bilinear functionals from the known representations of the linear operations.

Theorem 1. *The general form of the bilinear functional A in (E_1, E_2) is derived from the representation of the linear operation U from E_1 to $\bar{E}_2$ (or from E_2 to $\bar{E}_1$) and vice versa. The norm $\|A\|$ equals to the norm $\|U\|$.*

Proof. Let $A(f, g)$ be a bilinear functional, where $f \in E_1$, $g \in E_2$, then $|A(f, g)| \leqq \|A\|\cdot\|f\|\cdot\|g\|$. Therefore, if f is fixed, $|A(f, g)| \leqq M\|g\|$, that is $A(f, \cdot) \in \bar{E}_2$. Since $A(f, \cdot)$ is additive concerning f and

$$(1) \qquad \|A(f, \cdot)\| = \sup_{\|g\|=1} |A(f, g)| \leqq \|A\|\cdot\|f\| ,$$

$Uf \equiv A(f, \cdot)$ is the linear operation from E_1 to $\bar{E}_2$. Similarly $U'g \equiv A(\cdot, g)$ is the linear operation from E_2 to $\bar{E}_1$. Thus to any bilinear functional A in (E_1, E_2) corresponds a linear operation U from E_1 to $\bar{E}_2$ (or from E_2 to $\bar{E}_1$), and from (1)

(2) $$\| U \| \leqq \| A \| .$$

Conversely if a linear operation U from E_1 to $\bar{E}_2$ (or from E_2 to E_1) is given, then $Uf = U_f(\cdot) \in \bar{E}_2$. Put $U_f(g) = A(f, g)$, then this is a real number and is additive homogeneous concerning f and g. Moreover,

$$| A(f, g) | = | U_f(g) | \leqq \| U_f(\cdot) \| \cdot \| g \| \leqq \| U \| \cdot \| f \| \cdot \| g \| ,$$

therefore we get a linear functional A in (E_1, E_2), and

(3) $$\| A \| \leqq \| U \| .$$

In this way we get a one-to-one correspondence between the bilinear functionals and the linear operations, and from (1) and (3), it follows that

$$\| U \| = \| A \| .$$

Thus we get the theorem.

Representation of linear operations has been studied by many writers. For example, from the results of Phillips [2] and the Theorem 1, we get

Theorem 2. *Let $E_1 = L^p$, $(1 \leqq p \leqq \infty)$ and $E_2 = X$ (any Banach space), then the bilinear functional $A(f, g)$ in (E_1, E_2) is of the form:*

$$A(f, x) = \int f \, d\bar{x}(x, \tau) ,$$

where $f \in L^p$, $x \in X$, $\bar{x} \in \bar{X}$, $x(\cdot, \tau)$ is an abstract function such that $\bar{x}(\cdot, \tau) \in V^{p'}(\bar{X}, X)\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. The norm is given by

$$\| A \| = \| \bar{x}(\cdot, \tau) \| .$$

Proof is evident, since the linear operation from L^p to $\bar{X}$ is given by

$$U(f) = \int f \, d\bar{x}(\cdot, \tau)$$

where $\bar{x}(\cdot, \tau)$ is the above described abstract function. (cf. Phillips [2], Theorem 4.1.)

Now, if we take concrete Banach spaces, representation of linear operations and their norms are given by the usual Lebesgue (-Stieltjes) integrals in many cases. Especially if $(E_1, E_2) = (L, L^p)$, (C, L^p), (L, M) (M, M), (C, C), then the general forms of the bilinear functionals are obtained by the results of Gelfand [3] and Theorem 1, and are of the forms given by Prof. Izumi [1]. Moreover their norms are exactly determined. Thus, for instance,

Theorem 3. *If (E_1, E_2) be (L, L^p) $(p > 1$, then the bilinear functional $A(f, g)$ is written in the form:*

$$A(f, g) = \int_0^1 \int_0^1 f(t)\, g(u)\, \psi(t, u)\; dt\; du$$

where $\psi(t, u)$ is essentially bounded with respect to t in $(0, 1)$ and belongs to $L^{p'}$ with respect to $u(1/p+1/p' = 1)$. We have also

$$\|A\| = \operatorname{ess\cdot sup}_{0 \leq t \leq 1} \left(\int_0^1 |\psi(t, u)|^{p'} du \right)^{1/p'}.$$

Theorem 4. *The bilinear functional $A(f, g)$ on (C, L^p) $(p > 1)$ is written in the form:*

$$A(f, g) = \int_0^1 \int_0^1 f(t)\, g(u)\, d_t\, \psi(t, u) du$$

where $\psi(t, u)$ is of bounded variation with respect to t and belongs to $L^{p'}$ with respect to u, and

$$\|A\| = \left(\int_0^1 \left| \left(\int_0^1 |d_t\, \psi(t, u)| \right) \right|^{p'} du \right)^{1/p'}.$$

Remark: In the paper of Prof. Izumi [1], he stated the following theorem; the .bilinear functional $A(f, g)$ on (L^p, L^q), $(1 < p,\ q < \infty)$, is given by

$$A(f, g) = \int_0^1 \int_0^1 f(t)\, g(u)\, \psi(t, u)\; dt\; du,$$

where $\psi(t, u)$ belongs to $L^{p'}$ with respect to t and belongs to $L^{q'}$ with respect to u. But if we consider the linear operation $A(f, \cdot)$ or $A(\cdot\,, g)$ derived from the bilinear functional $A(f, g)$, then $A(f, \cdot)$ or $A(\cdot\,, g)$ is a completely continuous operation, therefore this result seems to be incomplete from Theorem 1.

References.

1) S. Izumi: On the bilinear functionals, Tôhoku Math. Journ., 42 (1936), 195-209.

2) R. S. Phillips: On the linear transformations, Trans. Amer. Math. Soc., 48 (1940), 516-541.

3) I. Gelfand: Abstrakte Funktionen und lineare Operatoren, Rec. Math., 4 (1938), 235-284.