151. *Probability-theoretic Investigations on Inheritance.*
V$_2$. *Brethren Combinations.*

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2$^{\text{bis}}$. Brethren combination with different fathers.

We shall now compare the probabilities σ_0's with the corresponding ones σ's. We have shown in (5.11) to (5.15) of IV that the inequality

(2.11) $$d(ij\,;ih,ih)\equiv\pi(ij\,;ih,ih)-\pi_0(ij\,;ih,ih)\geqq 0$$

is valid for any triple i, j, h which may coincide each other. Hence introducing a notation

(2.12) $$\delta(ih,\ ih)\equiv\sigma(ih,\ ih)-\sigma_0(ih,\ ih)$$

which corresponds to (5.10) of IV, we conclude immediately, in view of (1.2), that the similar inequality

(2.13) $$\delta(ih,\ ih)\geqq 0$$

remains valid for any pair i, h being admitted also to be coincident.

In other words, if we write down each table such that the types of the first and the second children are arranged in the same order, then every probability laid on the principal diagonal of the table on the brethren combination consisting of two children having their both parents in common is, in general, greater than the corresponding one of that on brethren combination consisting of two children having their mother alone in common. This statement is very reasonable, since it represents the fact that the tendency of resemblance between the types of brethren is stronger in the former case than in the latter.

In view of the inequalities obtained in a later part of § 5 of IV, the same is true also on phenotypes.

Corresponding to the exceptional cases stated in (5.18) and (5.19) of IV, we have at present, on the contrary, the inequalities

(2.14) $$\delta(ii,\ ij)=\ \delta(ij,\ ii)=\tfrac{1}{2}p_i^2p_j(1+p_i)-\tfrac{1}{2}p_i^2p_j(1+2p_i)$$
$$=-\tfrac{1}{2}p_i^3p_j\leqq 0 \qquad (i\neq j),$$

(2.15) $$\delta(ih,jh)=\delta(jh,ih)=\tfrac{1}{2}p_ip_jp_h(1+2p_h)-\tfrac{1}{2}p_ip_jp_h(1+4p_h)$$
$$=-p_ip_jp_h^2\leqq 0 \qquad (j,h\neq i\,;j\neq h).$$

Except those laid on the principal diagonal and those which have essentially be exhausted by (2.14) and (1.15), the remaining quantities σ's and σ_0's satisfy the equality

(2.16) $$\sigma(hk,\ fg) = \tfrac{1}{2}\sigma_0(hk,\ fg),$$

whence it follows, in particular,

(2.17) $$\delta(hk,\ fg) \leqq 0.$$

We thus conclude that every probability not laid on the principal diagonal of the table on brethren combination consisting of two children having their parents in common is, in general, always less than the corresponding one of that on brethren combination consisting of two children having their mother alone in common.

We now enter into the problem already announced at the end of the preceding chapter. The probability of random choice of two children of the types A_{hk} and A_{fg}, the order of which must be taken into account, is evidently given by the product

(2.18) $$\bar{A}_{hk}\bar{A}_{fg}.$$

Our main purpose is to compare the quantity (2.1) with the corresponding one (2.18), especially *to show the inequality*

(2.19) $$\sigma_0(ij,\ ij) \geqq \bar{A}_{ij}^2 \qquad (i, j=1, \ldots, m).$$

The last inequality will easily be verified. In fact, we obtain, in case $j = i$,

(2.20) $$\sigma_0(ii,\ ii) - \bar{A}_{ii}^2 = \tfrac{1}{2}p_i^3(1+p_i) - p_i^4 = \tfrac{1}{2}p_i^3(1-p_i) \geqq 0,$$

and, in case $j \neq i$,

(2.21) $$\begin{aligned}\sigma_0(ij,\ ij) - \bar{A}_{ij}^2 &= \tfrac{1}{2}p_ip_j(p_i+p_j+4p_ip_j) - 4p_i^2p_j^2 \\ &= \tfrac{1}{2}p_ip_j(p_i(1-p_j)+p_j(1-p_i)) \geqq 0.\end{aligned}$$

With respect to probabilities not laid on the principal diagonal, we first consider the exceptional cases where

(2.22) $$\sigma_0(ii,\ ij) - \bar{A}_{ii}\bar{A}_{ij} = \tfrac{1}{2}p_i^2p_j(1+2p_i) - 2p_i^3p_j = \tfrac{1}{2}p_i^2p_j(1-2p_i),$$

(2.23) $$\sigma_0(ij,\ ih) - \bar{A}_{ij}\bar{A}_{ih} = \tfrac{1}{2}p_ip_jp_h(1+4p_i) - 4p_i^2p_jp_h = \tfrac{1}{2}p_ip_jp_h(1-4p_i),$$

the suffices i, j, h being supposed to be different each other. The sign of the differences (2.22) and (2.23) is indeterminate. The sign of the former is positive or negative according that p_i is less or greater than 1/2 unless $p_ip_j = 0$. That of the latter is positive or negative according that p_i is less or greater than 1/4 unless $p_ip_jp_h = 0$.

Except only the cases essentially contained in (2.22) and (2.23), it is immediately seen that between every corresponding pair not laid on the principal diagonal the relation

(2.24) $$\sigma_0(hk,\ fg) = \tfrac{1}{2}\bar{A}_{hk}\bar{A}_{fg}$$

holds good, whence it follows, in particular, (2.25) $$\sigma_0(hk, fg) \leqq \bar{A}_{hk} A_{fg}$$

for such suffices h, k, f, g.

In conclusion, we investigate a correlation with respect to the resemblance between two children in case of MN blood type ($m = 2$). Suppose now that M, MN, N, are laid at the points 0, 1/2, 1 respectively on a coordinate axis. The centre of gravity of these points with respective weights equal to their frequencies lies at the point

(2.26) $$\mu \equiv s^2 \cdot 0 + 2st \cdot \tfrac{1}{2} + t^2 \cdot 1 = st + t^2 = t$$

which is regarded as the mean of distribution. The deviations of M, MN, N from the mean being

$$0-t=-t, \quad \tfrac{1}{2}-t=\tfrac{1}{2}(s-t), \quad 1-t=s$$

respectively, the *variance* of the distribution is given by

(2.27) $$\tau^2 \equiv s^2(1-t)^2 + 2st\,\tfrac{1}{4}(s-t)^2 + t^2 s^2 = \tfrac{1}{2}st,$$

whence the standard deviation is equal to $\tau = \sqrt{st/2}$.

Now, we shall calculate the correlation coefficients. If there is no blood connection between two children, then the co-variance becomes 0, whence it follows that the correlation coefficient vanishes. If two children are brethren having their mother alone in common, then the co-variance becomes $st/8$, whence the correlation, coefficient is equal to

(2.28) $$\tfrac{1}{8}\, st/\tau^2 = \tfrac{1}{4}.$$

If two children have their both parents in common, then the co-variance becomes $st/4$, whence the correlation coefficient is equal to

(2.29) $$\tfrac{1}{4}\, st/\tau^2 = \tfrac{1}{2}.$$

We thus conclude that, *in case of MN blood type, the correlation coefficients are* 0, 1/4, 1/2 *respectively, according that two children have no, only one, two among parents in common.*

3. Brethren combination concerning several children.

The results on mother-children combination with two children discussed in § 3 of IV, have been generalized in § 4 of IV to case of several children. Similar generalization can also be applied to the result discussed in § 1. The actual method will also be obvious.

We consider the sets of n children having both parents in common, and denote by

(3.1) $$\sigma(h_1k_1, \cdots, h_nk_n)$$

the probabilty of combination consisting of children with types $A_{h_1k_1}$, ..., $A_{h_nk_n}$, the order being taken into account, as a notation generalizing that in (1.1). Corresponding to (1.2), we get

$$\sigma(h_1k_1, \cdots, h_nk_n) = \sum_{i \leq j} \pi(ij; h_1k_1, \cdots, h_nk_n); \tag{3.2}$$

the summation in the right-hand side runs over all possible types A_{ij} of mother which are, as already noticed in § 4 of IV, equal to at most 4 in number.

By means of the results in § 4 of IV, we repeat the arguments in § 2 suitably modified. Abbreviation corresponding to (4.13) of IV will be used. The results are as follows:

$$\begin{aligned}\sigma(ii^n) &= \pi(ii; ii^n) + \sum_{j \neq i} \pi(ij; ii^n) \\ &= \frac{1}{2^{n-1}} p_i^3(1 + (2^{n-1}-1)p_i) + \frac{1}{4^{n-1}} p_i^2(1 + (2^{n-1}-1)p_i) \sum_{j \neq i} p_j \\ &= \frac{1}{4^{n-1}} p_i^2(1 + (2^{n-1}-1)p_i)^2,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\sigma(ij^n) &= \pi(ii; ij^n) + \pi(jj; ij^n) + \pi(ij; ij^n) + \sum_{h \neq i,j} (\pi(ih; ij^n) + \pi(jh; ij^n)) \\ &= \frac{1}{2^{n-1}} p_i^2 p_j(1 + (2^{n-1}-1)p_j) \\ &\quad + \frac{1}{2^{n-1}} p_j^2 p_i(1 + (2^{n-1}-1)p_i) \\ &\quad + \frac{1}{4^{n-1}} p_i p_j(p_i + p_j)(1 + (2^{n-1}-1)(p_i + p_j)) \\ &\quad + \frac{1}{4^{n-1}} p_i p_j(2 + (2^{n-1}-1)(p_i + p_j)) \sum_{h \neq i,j} p_h \\ &= \frac{1}{2^{2n-3}} p_i p_j(1 + (2^{n-1}-1)(p_i + p_j + 2^{n-1} p_i p_j)) \qquad (i \neq j),\end{aligned} \tag{3.4}$$

$$\sigma(ii^{n-\nu}, jj^\nu) = \pi(ij; ii^{n-\nu}, jj^\nu) = \frac{1}{4^{n-1}} p_i^2 p_j^2 \qquad (i \neq j;\ 0 < \nu < n), \tag{3.5}$$

$$\begin{aligned}\sigma(ii^{n-\nu}, ij^\nu) &= \pi(ii; ii^{n-\nu}, ij^\nu) + \pi(ij; ii^{n-\nu}, ij^\nu) + \sum_{h \neq i,j} \pi(ih; ii^{n-\nu}, ij^\nu) \\ &= \frac{1}{2^{n-1}} p_i^3 p_j + \frac{1}{4^{n-1}} p_i^2 p_j(1 + (2^{n-1}-1)p_i \\ &\qquad + (2^\nu - 1)p_j) + \frac{1}{4^{n-1}} p_i^2 p_j \sum_{h \neq i,j} p_h \\ &= \frac{1}{2^{2n-3}} p_i^2 p_j(1 + (2^{n-1}-1)p_i + (2^{\nu-1}-1)p_j) \\ &\qquad\qquad (i \neq j;\ 0 < \nu < n),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\sigma(ii^{n-\nu}, jh^\nu) &= \pi(ij; ii^{n-\nu}, jh^\nu) + \pi(ih; ii^{n-\nu}, jh^\nu) \\ &= \frac{1}{4^{n-1}} p_i^2 p_j p_h + \frac{1}{4^{n-1}} p_i^2 p_h p_j = \frac{1}{2^{2n-3}} p_i^2 p_j p_h \\ &\qquad\qquad (j, h \neq i;\ j \neq h;\ 0 < \nu < n),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\sigma(ij^{n-\nu}, ih^\nu) &= \pi(ii; ij^{n-\nu}, ih^\nu) + \pi(ij; ij^{n-\nu}, ih^\nu) + \pi(ih; ij^{n-\nu}, ih) \\ &\quad + \pi(jh; ij^{n-\nu}, ih^\nu) + \sum_{k \neq i,j,h} \pi(ik; ij^{n-\nu}, ih^\nu) \\ &= \frac{1}{2^{2n-3}} p_i p_j p_h(1 + 2^{n-1} p_i) \qquad (j, h \neq i;\ j \neq h;\ 0 < \nu < n),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\sigma(ij^{n-\nu}, hk^{\nu}) &= \pi(ih; ij^{n-\nu}, hk^{\nu}) + \pi(ik; ij^{n-\nu}, hk^{\nu}) \\ &\quad + \pi(jh; ij^{n-\nu}, hk^{\nu}) + \pi(jk; ij^{n-\nu}, hk^{\nu}) \\ &= \frac{1}{4^{n-2}} p_i p_j p_h p_k \qquad (h, k \neq i, j;\ i \neq j;\ h \neq k;\ 0<\nu<n),\end{aligned} \tag{3.9}$$

$$\sigma(ii^{n-\mu-\nu}, jj^{\mu}, ij^{\nu}) = \pi(ij; ii^{n-\mu-\nu}, jj^{\mu}, ij^{\nu}) = \frac{1}{2^{2n-\nu-2}} p_i^2 p_j^2 \qquad (i \neq j;\ 0<\mu\leqq\mu+\nu<n), \tag{3.10}$$

$$\begin{aligned}\sigma(ii^{n-\mu-\nu}, ij^{\mu}, ih^{\nu}) &= \pi(ij; ii^{n-\mu-\nu}, ij^{\mu}, ih^{\nu}) + \pi(ih; ii^{n-\mu-\nu}, ij^{\mu}, ih^{\nu}) \\ &= \frac{1}{2^{2n-3}} p_i^2 p_j p_h \quad (j, h \neq i;\ j \neq h;\ 0<\mu<\mu+\nu<n),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\sigma(ii^{n-\mu-\nu}, ij^{\mu}, jh^{\nu}) &= \pi(ij; ii^{n-\mu-\nu}, ij^{\mu}, jh^{\nu}) + \pi(ih; ii^{n-\mu-\nu}, ij^{\mu}, jh^{\nu}) \\ &= \frac{1}{2^{2n-3}} p_i^2 p_j p_h \quad (j, h \neq i;\ j \neq h;\ 0<\mu<\mu+\nu<n),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\sigma(ij^{n-\mu-\nu}, ih^{\mu}, jh^{\nu}) &= \pi(ij; ij^{n-\mu-\nu}, ih^{\mu}, jh^{\nu}) \\ &\quad + \pi(ih; ij^{n-\mu-\nu}, ih^{\mu}, jh^{\nu}) + \pi(jh; ij^{n-\mu-\nu}, ih^{\mu}, jh^{\nu}) \\ &= \frac{1}{2^{2n-3}} p_i p_j p_h (p_i + p_j + p_h) \\ &\qquad (j, h \neq i;\ j \neq h;\ 0<\nu<\mu+\nu<n),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\sigma(ih^{n-\mu-\nu}, ik^{\mu}, jh^{\nu}) &= \pi(ij; ih^{n-\mu-\nu}, ik^{\mu}, jh^{\nu}) \\ &\quad + \pi(hk; ih^{n-\mu-\nu}, ik^{\mu}, jh^{\nu}) \\ &= \frac{1}{2^{2n-3}} p_i p_j p_h p_k \\ &\qquad (h, k \neq i, j;\ i \neq j;\ h \neq k;\ 0<\nu<n)\end{aligned} \tag{3.14}$$

$$\begin{aligned}\sigma(ii^{n-\lambda-\mu-\nu}, ij^{\lambda}, ih^{\mu}, jh^{\nu}) &= \pi(ij; ii^{n-\lambda-\mu-\nu}, ij^{\lambda}, ih^{\mu}, jh^{\nu}) \\ &\quad + \pi(ih; ii^{n-\lambda-\mu-\nu}, ij^{\lambda}, ih^{\mu}, jh^{\nu}) \\ &= \frac{1}{2^{2n-3}} p_i^2 p_j p_h \\ &\qquad (j, h \neq i;\ j \neq h;\ 0<\mu+\nu\leqq\lambda+\mu+\nu<n),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\sigma(ih^{n-\lambda-\mu-\nu}, ik^{\lambda}, jh^{\mu}, jk^{\nu}) &= \pi(ij; ih^{n-\lambda-\mu-\nu}, ik^{\lambda}, jh^{\mu}, jk^{\nu}) \\ &\quad + \pi(hk; ih^{n-\lambda-\mu-\nu}, ik^{\lambda}, jh^{\mu}, jk^{\nu}) \\ &= \frac{1}{2^{2n-3}} p_i p_j p_h p_k \\ &\qquad (h, k \neq i, j;\ i \neq j;\ h \neq k;\ 0<\mu+\nu\leqq\lambda+\mu+\nu\leqq n).\end{aligned} \tag{3.16}$$

Thus, all the possible cases have essentially been worked out.

If we sum up all the possible probabilities with respect to one of n children, then we get, of course, the corresponding probaility with $n-1$ children. For instance, summing up the probabilities with fixed types, all being coincident with A_{ii} say, of the second to the last children over the types of the first child, then we obtain

$$\sigma(ii^n)+\sum_{j\neq i}(\sigma(jj,\,ii^{n-1})+\sigma(ij,\,ii^{n-1}))+\sum_{j,h\neq i;\,j<h}\sigma(jh,\,ii^{n-1})$$
$$=\frac{1}{4^{n-1}}\,p_i^2(1+(2^{n-1}-1)p_i)^2$$
$$+\sum_{j\neq i}\left(\frac{1}{4^{n-1}}p_i^2p_j^2+\frac{1}{2^{2n-3}}\,p_i^2p_j(1+(2^{n-1}-1)p_i)\right)+\sum_{j,\,h\neq i,\,j<h}\frac{1}{2^{2n-3}}\,p_i^2p_jp_h$$
$$=\frac{1}{4^{n-1}}\,p_i^2((1+(2^{n-1}-1)p_i)^2+2(1+(2^{n-1}-1)p_i)(1-p_i)+(1-p_i)^2)$$
$$=\frac{1}{4^{n-2}}\,p_i^2(1+(2^{n-2}-1)p_i)^2,$$

the value which coincides just with (3.3), n being replaced by $n-1$.

The procedure of passage to results on phenotypes is usual. But, in each concrete case, direct calculations by making use of the corresponding mother-children combination will imply the results more immediately.