3. On Rings of Operators of Infinite Classes.

By Haruo SUNOUCHI.

Mathematical Institute, Tohoku University, Sendai.

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Let $\boldsymbol{M}$ be a ring of operators in a Hilbert space $\boldsymbol{H}$ in the sense of J. von Neumann [3], and denote the center of $\boldsymbol{M}$ by $\boldsymbol{M}^{\natural}$. Recently J. Dixmier has proved the following theorem [1; Theorems 10 and 11]:

Theorem of Dixmier. *If $\boldsymbol{M}$ is of finite class, then there exists a mapping $A \rightarrow A^{\natural}$ of $\boldsymbol{M}$ on $\boldsymbol{M}^{\natural}$ possessing the following properties:*

(1) *If* $A \in \boldsymbol{M}^{\natural}$, $A^{\natural} = A$,

(2) $(\lambda A)^{\natural} = \lambda A^{\natural}$,

(3) $(A + B)^{\natural} = A^{\natural} + B^{\natural}$,

(4α) $(AB)^{\natural} = (BA)^{\natural}$,

(4β) $(AB)^{\natural} = AB^{\natural}$ *if* $A \in \boldsymbol{M}^{\natural}$,

(5α) *If* $A \in \boldsymbol{M}_s$ *and* $A \geqq 0$, *then* $A^{\natural} \in \boldsymbol{M}_s$ *and* $A^{\natural} \geqq 0$,

(5β) *If* $A \in \boldsymbol{M}_s$, $A \geqq 0$ *and* $A^{\natural} = 0$, *then* $A = 0$,

(6) $(A^*)^{\natural} = (A^{\natural})^*$.

Furthermore, if there exists a mapping $A \rightarrow \varphi(A)$ of $\boldsymbol{M}$ on $\boldsymbol{M}^{\natural}$ with the properties (1) (2) (3) (4α) *and* (5α), *then* $\varphi(A) = A^{\natural}$ *for all* $A \in \boldsymbol{M}$.

The present paper is a continuation of the one of Dixmier [1], and our object is to generalise the notion of his $\natural$-operation for the rings of operators of infinite classes. If $\boldsymbol{M}$ is a factor, our results include the one of Neumann [4].

We shall use the usual definitions and notations in the theory of rings of operators without any explanation, and the results of Dixmier will be assumed. The reader is reffered to [1] or [3].

1. Following [1] and [3], we shall say that a projection $E \in \boldsymbol{M}$ is *finite* if, for any projection $F \in \boldsymbol{M}$ $E \sim F$, $F \leqq E$ implies $F = E$, and *infinite* if this is not the case. If the unit element I is finite, then we say $\boldsymbol{M}$ is *of finite class*, and otherwise $\boldsymbol{M}$ is *of infinite class.*

Consider those operators $A \in \boldsymbol{M}$ which are permutable with a projection $E \in \boldsymbol{M}$ and form their parts in E, $A_{(E)}$. Denote the set of all those $A_{(E)}$ ($A \in \boldsymbol{M}$, and permutable with E) by $\boldsymbol{M}_{(E)}$. We say $A \in \boldsymbol{M}$ is *contained* in E if $AE = EA = A$. Then obviously $\boldsymbol{M}_{(E)}$ is a ring of operators in $\boldsymbol{EH}$ and $(\boldsymbol{M}_{(E)})^{\natural} = (\boldsymbol{M}^{\natural})_{(E)}$ [3; Lemmas 11. 3. 2 and 11. 3. 4].

By the Dixmier theorem, there exists a mapping $A \rightarrow A^{\natural_E'}$ of $\boldsymbol{M}_{(E)}$ on $\boldsymbol{M}^{\natural}_{(E)}$ for any finite projection $E \in \boldsymbol{M}$. We now consider to construct the mapping $A \rightarrow A^{\natural_E}$ of $\boldsymbol{M}_{(E)}$ in $\boldsymbol{M}^{\natural}$ with the conditions (1)—(6).

By *central envelope* of a projection $E \in \boldsymbol{M}$ we mean the least central projection containing E and denote it by Z. This notion is equivalent to the $[E^{\boldsymbol{M}}]$ of Dixmier, therefore Z is the least upper bound of F, which are equivalent to E.

Lemma 1. *There exists a one-to-one mapping Θ of $\boldsymbol{M}^{\natural}_{(E)}$ onto $\boldsymbol{M}^{\natural}_{(Z)}$ which has the following properties:*

(1) *If* $A \in \boldsymbol{M}^{\natural}_{(Z)}$, $\Theta(EA) = A$,

(2) *Θ is an algebraic ring-isomorphism,*

(3) *If A is self-adjoint and $A \geqq 0$, then $\Theta(A)$ is self-adjoint and $\Theta(A) \geqq 0$,*

(4) *Θ is an isometry.*

Proof. First we remark that for any projection $P \in \boldsymbol{M}^{\natural}_{(Z)}$, $EP=0$ implies $P=0$. In fact, $EP=0$ implies $E \leqq Z-P \in \boldsymbol{M}^{\natural}_{(Z)} \leqq \boldsymbol{M}^{\natural}$, hence $P=0$ by the definition of Z. Therefore there exists a one-to-one correspondence between the projections in $\boldsymbol{M}^{\natural}_{(E)}$ and $\boldsymbol{M}^{\natural}_{(Z)}$. Denote this by Θ. If we define, for any $A = \int \lambda \, dP_\lambda$ in $\boldsymbol{M}^{\natural}_{(E)}$, $\Theta(A) = \int \lambda \, d\,\Theta(P_\lambda)$, then we obtain the required Θ.

Let a projection $E \in \boldsymbol{M}$ be finite. We shall define the operation $\natural_E$ by

$$A^{\natural'_E} = \Theta(A^{\natural'_E})$$

for any $A \in \boldsymbol{M}$ contained in E. Then by the properties of $A^{\natural'_E}$ and Θ we obtain the following

Theorem 1. *There exists a mapping $A \rightarrow A^{\natural_E}$ in $\boldsymbol{M}^{\natural}$ satisfying the conditions* (2) (3) (4α) (4β) (5α) (5β) (6), *and this mapping is locally uniformly continuous.*

Remark. It is easily seen that the finite central projections have the maximal one, therefore $\boldsymbol{M}$ is decomposed into the direct sum of three rings of operators, $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{M}_3$, say; $\boldsymbol{M}_1$ is of finite class, $\boldsymbol{M}_2$ is the one, in which every central projection is infinite, but there exists a finite projection, and $\boldsymbol{M}_3$ is in the other case. We say $\boldsymbol{M}_3$ is *of the purely infinite class.* The first case is reduced to the Dixmier theorem and in the third case our method is not available. Therefore we shall discuss only the second case (except for the last section). In this case it is known that $A^{\natural}$ cannot be defined for $A \in \boldsymbol{M}^{\natural}_2$ [1; Theorem 14], so the condition (1) is vacuous.

2. The above obtained $\natural_E$-operation depends on E. We shall next study this point. First we note the relation between $A^{\natural_E}$ and $A^{\natural}$ of the finite class.

Lemma 2. *Let $\boldsymbol{M}$ be of finite class. If $E \in \boldsymbol{M}$ be any projection and Z denotes its central envelope, then* Range $(E^{\natural})=Z$ *and* $E^{\natural}$ *has an inverse* $(E^{\natural})^{-1}$ *in* $\boldsymbol{M}^{\natural}_{(Z)}$.

Proof. By [1; Lemmas 4.10. and 6.4], E can be written in the form $E=\sum_{\alpha \in I} \oplus P_{\alpha}$, where P_{α} are mutually orthogonal and simple (cf [1; Definition 4.2]). The $\natural$-operation is strongly continuous in the unit sphere [1; Theorem 17], hence $E^{\natural}=\sum P_{\alpha}^{\natural}=\sum \frac{1}{n_{\alpha}} Z_{\alpha}$, where Z_{α} is the central envelope of P_{α} [1; Theorem 10 (7)]. Therefore Range $(E^{\natural}) \geqq Z$ is clear. Let us prove the inverse implication. It is sufficient to prove that for any finite subset J of I we have Range $(\sum_{\alpha \in J} Z_{\alpha}) \leqq Z$. It follows easily by Lemma 6.6 of [1].

The existence of $(E^{\natural})^{-1}$ is evident if we remark that $\boldsymbol{M}^{\natural}_{(Z)}$ can be represented by the space of continuous functions.

Theorem 2. *Let $\boldsymbol{M}$ be of finite class, and $A \in \boldsymbol{M}$ be contained in a projection E. Then we have*

$$A^{\natural_E \prime}=E\left((E^{\natural})^{-1} A^{\natural}\right)$$

or

$$A^{\natural_E}=(E^{\natural})^{-1} A^{\natural},$$

where $A^{\natural}$ is the one given by the Dixmier theorem.

Proof. The existence of $(E^{\natural})^{-1}$ in $\boldsymbol{M}^{\natural}_{(Z)}$ was shown in Lmema 2, and $A^{\natural}$ is contained in $\boldsymbol{M}^{\natural}_{(Z)}$, hence the above expression has the meaning. Now $E((E^{\natural})^{-1}A^{\natural})$ belong to $\boldsymbol{M}^{\natural}_{(E)}$ and satisfy all the properties of the $\natural_E'$-operation. Because, if $A \in \boldsymbol{M}^{\natural}_{(E)}$, then it has the form $A=EB$, $B \in \boldsymbol{M}^{\natural}$, therefore $A^{\natural}=E^{\natural}B$, and $E((E^{\natural})^{-1}A^{\natural})=EB=A$, therefore condition (1) is satisfied, and the other conditions are evident. Hence by the unicity of the $\natural_E'$-operation we obtain the result.

3. If projections E, $F \in \boldsymbol{M}$ be finite, then it is well-known that $P=E \cup F$ is also finite (for example, cf. [2; theorem 6.2]). If A is contained in $E \cap F$, then we can define $A^{\natural_E}$, $A^{\natural_F}$ and $A^{\natural_P}$. We shall study the relations between them.

Lemma 3. *Let a projection $P \in \boldsymbol{M}$ is finite and $A \in \boldsymbol{M}$ is contained in $E \leqq P$, then the above $A^{\natural_E}$ and $A^{\natural_P}$ is related by*

$$A^{\natural_P}=E^{\natural_P} A^{\natural_E}.$$

Proof. We consider the central envelopes Z_E and Z_P. As $E \in \boldsymbol{M}_{(P)}$, we consider also the central envelope $Z_{E,P}$ of E in $\boldsymbol{M}_{(P)}$. Θ_E, Θ_P and $\Theta_{E,P}$ denote the mappings analogous to the Θ constructed in lemma 1, that is, Θ_E is a mapping from $\boldsymbol{M}^{\natural}_{(E)}$ to $\boldsymbol{M}^{\natural}_{(Z_E)}$, Θ_P from $\boldsymbol{M}^{\natural}_{(P)}$ to $\boldsymbol{M}^{\natural}_{(Z_P)}$ and $\Theta_{E,P}$ from $\boldsymbol{M}^{\natural}_{(E)}$ to $\boldsymbol{M}^{\natural}_{(Z_{E,P})}$. Then we obtain from the theorem 2,

$$A^{\natural P} = \Theta_P E^{\natural' P} \Theta_{E,P} \Theta_E^{-1} A^{\natural E}$$
$$= E^{\natural P} \Theta_P \Theta_{E,P} (E(PA^{\natural E}))$$
$$= E^{\natural P} \Theta_P (PA^{\natural E}) = E^{\natural P} A^{\natural E}.$$

Theorem 3. *Let the projections E, $F \in \boldsymbol{M}$ be finite.* If $A \in \boldsymbol{M}$ *is contained in $E \frown F$, we can define $A^{\natural E}$ and $A^{\natural F}$. Then we have*

$$A^{\natural F} = (F^{\natural P})^{-1} E^{\natural P} A^{\natural E},$$

where $B^{\natural P}$ is defined by any finite projection $P \in \boldsymbol{M}$ containing E and F.

Especially, if $E \sim F$, then we obtain $A^{\natural F} = A^{\natural E}$.

Proof. Evident from Lemma 3 and the fact that the above expression has the meaning, because $A^{\natural E}$, $A^{\natural F}$ are contained in the central envelope of $E \frown F$.

4. We shall extend the notion of the $\natural$-operation of Theorem 1. Let a projection $E \in \boldsymbol{M}$ be finite, then any projection $P \in \boldsymbol{M}_{(Z)}$ can be written in the following form:

1° $P = \sum \oplus P_\alpha$, where P_α are mutually orthogonal and $P_\alpha \lesssim E$, or

2° $P = \sum_{i=1}^{N} \oplus P_i$ where P_i are mutually orthogonal and $P_i \lesssim E$, in some way.

In the case 2°, we shall say P is *E-finite*. Generally if $A \in \boldsymbol{M}$ is contained in some E-finite projection, then we shall say A is *E-finite*. If we define $P^{\natural E}$ for any E-finite projection P by

$$P^{\natural E} = \sum_{i=1}^{N} P_i^{\natural E},$$

and for any E-finite $A = \int \lambda \, dP_\lambda$,

$$A^{\natural E} = \int \lambda \, dP_\lambda^{\natural E},$$

then we obtain the $A^{\natural E}$ for any E-finite $A \in \boldsymbol{M}$.

Now we shall consider the arbitrary ring of operators $\boldsymbol{M}$. As we remarked above, $\boldsymbol{M}$ is a direct sum of $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{M}_3$. If the projections E, $F \in \boldsymbol{M}$ be non-comparable, then the central envelopes Z_E and Z_F are mutually orthogonal. Therefore, for some system of mutually non-comparable finite projections $E_\alpha \in \boldsymbol{M}_2$, the corresponding central envelopes span the unit element I_2 of $\boldsymbol{M}_2$. And any finite projection in $\boldsymbol{M}$ is a direct sum of a projection in $\boldsymbol{M}_1$ and some E_α-finite projections. If A is contained in some finite projection, we shall say A is *finite*. For any finite $A \in \boldsymbol{M}$ we can define a $\natural$-operation by a sum of $A^{\natural}_{(I_1)}$ and $A^{\natural E_\alpha}_{(Z E\alpha)}$. Then we obtain our final

Theorem 4. *Let $\boldsymbol{M}$ be a ring of operators. For any finite $A \in \boldsymbol{M}$, we can define a mapping $A^\natural \in \boldsymbol{M}^\natural$ satisfying the all required properties.*

If we define the $A^{\natural_1}$ by another system F_β, and if $Z_{E\alpha}$ and $Z_{F\beta}$ have a comparable part, then it is related by Theorem 3 in that part.

Proof. The first part is obvious.

If $Z_{E\alpha}$ and $Z_{F\beta}$ have a comparable part, then there exist $E'_\alpha \sim E_\alpha$ and $F'_\beta \sim F_\beta$ such that $E'_\alpha \cap F'_\beta$ is not empty. By this fact and the above described definition of $A^{\natural_{E\alpha}}$, $A^{\natural_{F\beta}}$, we can apply the theorem 3 for any finite A contained in $Z_{E\alpha}$ and $Z_{F\beta}$.

References.

1) J. Dixmier: Les anneaux d'opérateurs de classe finie. Ann. L'Ecole Norm. Sup. 66 (1950) 209-251.

2) I. Kaplansky: Projections in Banach algebras. Ann. of Math. 53 (1951) 235-249.

3) F.J. Murray and J. von Neumann: On rings of operators. Ann. of Math 37 (1936) 116-229.

4) J. von Neumann: On rings of operators III. Ann. of Math. 41 (1940) 96-161.

Added in proof. The statesments of **4** are not complete. This part will be discussed in detail in a next paper.