54. Probability-theoretic Investigations on Inheritance. IX$_4$. Non-Paternity Concerning Mother-Children Combinations.

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7$^{\text{bis}}$. Probability against at least one child.

The results on partial sums with respect to (7.2) can also be obtained in a direct manner by means of the following table concerning the set of deniable types of a man and its probability (7.1) against each triple.

Mother	1st child \ 2nd child	A_{ii}	A_{ih}	A_{ik}
A_{ii}	A_{ii}	$A_{ab}(a, b\neq i)$ $(1-p_i)^2$	$A_{ab}(\neq A_{ih})$ $1-2p_ip_h$	$A_{ab}(\neq A_{ik})$ $1-2p_ip_k$
	A_{ih}	$A_{ab}(\neq A_{ih})$ $1-2p_ip_h$	$A_{ab}(a, b\neq h)$ $(1-p_h)^2$	$A_{ab}(\neq A_{hk})$ $1-2p_hp_k$
	A_{ik}	$A_{ab}(\neq A_{ik})$ $1-2p_ip_k$	$A_{ab}(\neq A_{hk})$ $1-2p_hp_k$	$A_{ab}(a, b\neq k)$ $(1-p_k)^2$

Mother	1st child \ 2nd child	A_{ii}	A_{jj}	A_{ij}	A_{ih} / A_{jh}	A_{ik} / A_{jk}
A_{ij}	A_{ii}	$A_{ab}(a, b\neq i)$ $(1-p_i)^2$	$A_{ab}(\neq A_{ij})$ $1-2p_ip_j$	$A_{ab}(a, b\neq i)$ $(1-p_i)^2$	$A_{ab}(\neq A_{ih})$ $1-2p_ip_h$	$A_{ab}(\neq A_{ik})$ $1-2p_ip_k$
	A_{jj}	$A_{ab}(\neq A_{ij})$ $1-2p_ip_j$	$A_{ab}(a, b\neq j)$ $(1-p_j)^2$	$A_{ab}(a, b\neq j)$ $(1-p_j)^2$	$A_{ab}(\neq A_{jh})$ $1-2p_jp_h$	$A_{ab}(\neq A_{jk})$ $1-2p_jp_k$
	A_{ij}	$A_{ab}(a, b\neq i)$ $(1-p_i)^2$	$A_{ab}(a, b\neq j)$ $(1-p_j)^2$	$A_{ab}(a, b\neq i, j)$ $(1-p_i-p_j)^2$	$A_{ab}(\neq A_{ih}, A_{jh})$ $1-2(p_i+p_j)p_h$	$A_{ab}(\neq A_{ik}, A_{jk})$ $1-2(p_i+p_j)p_k$
	A_{ih} or A_{jh}	$A_{ab}(\neq A_{ih})$ $1-2p_ip_h$	$A_{ab}(\neq A_{jh})$ $1-2p_jp_h$	$A_{ab}(\neq A_{ih}, A_{jh})$ $1-2(p_i+p_j)p_h$	$A_{ab}(a, b\neq h)$ $(1-p_h)^2$	$A_{ab}(\neq A_{hk})$ $1-2p_hp_k$
	A_{ik} or A_{jk}	$A_{ab}(\neq A_{ik})$ $1-2p_ip_k$	$A_{ab}(\neq A_{jk})$ $1-2p_jp_k$	$A_{ab}(\neq A_{ik}, A_{jk})$ $1-2(p_i+p_j)p_k$	$A_{ab}(\neq A_{hk})$ $1-2p_hp_k$	$A_{ab}(a, b\neq k)$ $(1-p_k)^2$

We first derive the relations concerning (7.5) which correspond to (2.6) to (2.10) or (4.13) to (4.17). The results are as follows:

$$(7.10)\quad \tilde{J}(ii;ii)=\tfrac{1}{2}p_i^3(2-2(1+S_2)p_i-p_i^2+3p_i^3),$$

$$(7.11)\quad \tilde{J}(ii;ih)=\tfrac{1}{2}p_i^2p_h(2-2(1+S_2)p_h-p_h^2+3p_h^3);$$

$$(7.12)\quad \tilde{J}(ij;ii)=\tfrac{1}{4}p_i^2p_j(4-4(1+S_2)p_i-2p_i(p_i+p_j)+6p_i^3+p_ip_j(p_i+2p_j)),$$

$$\tilde{J}(ij;ij)=\tfrac{1}{4}p_ip_j(4(p_i+p_j)-4(1+S_2)(p_i^2+p_j^2)-4(1+2S_2)p_ip_j \tag{7.13}$$
$$-2(p_i^3+p_j^3)-5p_ip_j(p_i+p_j)$$
$$+6(p_i^4+p_j^4)+13p_ip_j(p_i^2+p_j^2)+14p_i^2p_j^2),$$

$$\tilde{J}(ij;ih)=\tfrac{1}{4}p_i^2p_jp_h(2-2(1+S_2+p_ip_j)p_h-p_h^2+3p_h^3). \tag{7.14}$$

Thus, all the possible cases have essentially been worked out. The results corresponding to (2.11) to (2.15) or (4.18) to (4.22) become now as follows:

$$\sum_{i=1}^{m}\tilde{J}(ii;ii)=S_3-S_4-\tfrac{1}{2}S_5-S_2S_4+\tfrac{3}{2}S_6, \tag{7.15}$$

$$\sum_{i=1}^{m}\sum_{h\neq i}\tilde{J}(ii;ih)=S_2-S_3-S_2^2+S_4-\tfrac{1}{2}S_2S_3+\tfrac{1}{2}S_5-S_2^3+\tfrac{5}{2}S_2S_4-\tfrac{3}{2}S_6; \tag{7.16}$$

$$\sum_{i,j}{}'(\tilde{J}(ij;ii)+\tilde{J}(ij;jj)) \tag{7.17}$$
$$=S_2-2S_3+\tfrac{1}{2}S_4-\tfrac{3}{2}S_2S_3+\tfrac{5}{2}S_5+\tfrac{1}{2}S_3^2+\tfrac{5}{4}S_2S_4-\tfrac{9}{4}S_6,$$

$$\sum_{i,j}{}'\tilde{J}(ij;ij) \tag{7.18}$$
$$=S_2-2S_3-\tfrac{1}{2}S_2^2+S_4-\tfrac{9}{4}S_2S_3+\tfrac{13}{4}S_5-S_2^3+\tfrac{7}{4}S_3^2+\tfrac{21}{4}S_2S_4-\tfrac{13}{2}S_6,$$

$$\sum_{i,j}{}'\sum_{h\neq i,j}(\tilde{J}(ij;ih)+\tilde{J}(ij;jh)) \tag{7.19}$$
$$=1-4S_2+\tfrac{7}{2}S_3+\tfrac{1}{2}S_4+\tfrac{5}{2}S_2S_3-4S_5-\tfrac{1}{2}S_2S_4+S_6.$$

The sums of (7.15) to (7.16) and of (7.17) to (7.19) are then

$$S_2(1-S_2-\tfrac{1}{2}S_3-S_2^2+\tfrac{3}{2}S_4), \tag{7.20}$$

$$1-2S_2-\tfrac{1}{2}S_3-\tfrac{1}{2}S_2^2+2S_4-\tfrac{5}{4}S_2S_3+\tfrac{7}{4}S_5-S_2^3+\tfrac{9}{4}S_3^2+6S_2S_4-\tfrac{31}{4}S_6, \tag{7.21}$$

respectively. The sum of the last two sub-probabilities yields the whole probability of the present non-paternity problem

$$\tilde{J}=1-S_2-\tfrac{1}{2}S_3-\tfrac{3}{2}S_2^2+2S_4-\tfrac{7}{4}S_2S_3+\tfrac{7}{4}S_5 \tag{7.22}$$
$$-2S_2^3+\tfrac{9}{4}S_3^2+\tfrac{15}{2}S_2S_4-\tfrac{31}{4}S_6.$$

It is a matter of course, as previously noticed, that the last result coincides just with the one in (7.5).

The corresponding quantity in mixed case becomes

$$\tilde{J}'=2I'-J'=1-S_2'-\tfrac{1}{2}S_3'-\tfrac{1}{2}(2S_2'^2+S_{1,1}^{\,2})+\tfrac{1}{2}(3S_4'+S_{2,2}) \tag{7.23}$$
$$-\tfrac{1}{4}(-2S_2'S_{1,2}+7S_{1,1}S_{1,2}+2S_2'S_{2,1})+\tfrac{7}{4}S_{2,3}$$
$$-2S_2'^2S_2+\tfrac{9}{4}S_{1,2}^{\,2}+\tfrac{1}{2}(11S_{1,1}S_{1,3}+4S_2'S_{2,2})-\tfrac{31}{4}S_{2,4}.$$

In conclusion, it would be noticed that the non-paternity proof is never possible at all except the cases deniable against at least one child, and consequently that the probability of an event that the non-paternity proof is impossible against any child is expressed as the complementary probability of the exceptional cases; namely, it is equal to $1-\tilde{J}$.

8. Maximizing distributions.

In the preceding sections of this chapter we have derived the explicit expressions for probabilities of various events. We shall now determine the distribution of genes which maximizes the respective probability. As a model, we begin with a concrete example, MN human blood type, which is regarded as the simplest case, i.e., $m=2$, of the general development.

The whole probability of proving non-paternity against second child at any rate, i.e., I_{MN}, coincides quite with the probability P_{MN} previously given in (5.1) of VIII, and hence may be omitted.

The whole probability against both children is equal to

$$J_{MN}=\tfrac{1}{4}st(1+2st)(2-3st). \tag{8.1}$$

Differentiating this expression by $x=st$, we get the derivative

$$dJ_{MN}/dx=\tfrac{1}{2}(1+x-9x^2)=\tfrac{1}{2}((1-4x)(1+5x)+11x^2),$$

which remains always positive in the interval $0\leqq x\leqq 1/4$. Hence, the maximizing distribution is given by

$$s=t=1/2;\quad \overline{M}=\overline{N}=1/4,\quad \overline{MN}=1/2 \tag{8.2}$$

and the maximum value by

$$(J_{MN})^{\max}=15/128=0.1172. \tag{8.3}$$

The whole probability against a distinguished child alone is equal to

$$I_{MN}-J_{MN}=\tfrac{1}{4}st(2-5st+6s^2t^2). \tag{8.4}$$

The derivative of (8.4) with respect to $x=st$ is

$$d(I_{MN}-J_{MN})/dx=\tfrac{1}{2}(1-5x+9x^2)=\tfrac{1}{2}((1-4x)(1-x)+5x^2),$$

which remains also always positive for $0\leqq x\leqq 1/4$. Hence, the maximizing distribution is again given by (8.2) and the maximum value by

$$(I_{MN}-J_{MN})^{\max}=9/128=0.0703. \tag{8.5}$$

Consequently, it is noticed that the relation

$$(I_{MN}-J_{MN})^{\max}=(I_{MN})^{\max}-(J_{MN})^{\max} \tag{8.6}$$

holds good. Namely, the quantity J_{MN} increases with st, but the quantity I_{MN} increases more rapidly so that the difference $I_{MN}-J_{MN}$ attains its maximum for the same distribution (8.2) for which the maximum of I_{MN} as well as J_{MN} is attained.

From the same reason, we conclude that the probability against at least one child, given by

(8.7) $$\tilde{J}_{MN}=2I_{MN}-J_{NM}=\tfrac{3}{4}st(2-3st+2s^2t^2),$$

attains its maximum again for the distribution (8.2); the maximum value is equal to

(8.8) $$(\tilde{J}_{MN})^{\max}=33/128=0.2598.$$

As a further illustrative example, we consider *ABO* blood type, in which a recessive gene appears. The probability J_{ABO} in (4.26) may be regarded, in view of the identity $p+q+r=1$, as a function of two independent variables p and q. In order to determine the maximizing distribution, we differentiate this function with respect to p and q, obtaining a system of equations $0=\partial J_{ABO}/\partial p=\partial J_{ABO}/\partial q$. Thus, we get the maximizing distribution

(8.9) $$p=q=0.2214, \qquad r=0.5572;$$
$$\bar{O}=0.3105, \qquad \bar{A}=\bar{B}=0.2957, \qquad \overline{AB}=0.0981;$$

the extremal values of p and q coinciding each other are both a root, contained in the interval $0<x<1/2$, of the quintic equation

(8.10) $$348x^5-610x^4+408x^3-105x^2+2=0.$$

The maximum of J_{ABO} is equal to

(8.11) $$(J_{ABO})^{\max}=0.1250.$$

For the whole probability against a distinguished child alone, i.e.,

(8.12) $$I_{ABO}-J_{ABO}=\tfrac{1}{2}p(1-p)^5+\tfrac{1}{2}q(1-q)^5+\tfrac{1}{4}pqr^2(8-5r-7r^2),$$

we get, in a similar manner, the maximizing distribution

(8.13) $$p=q=0.2206, \qquad r=0.5588;$$
$$\bar{O}=0.3123, \qquad \bar{A}=\bar{B}=0.2952, \qquad \overline{AB}=0.0973;$$

p and q being a root of the quintic equation

(8.14) $$348x^5-710x^4+472x^3-141x^2+24x-2=0.$$

The maximum of $I_{ABO}-J_{ABO}$ is equal to

(8.15) $$(I_{ABO}-J_{ABO})^{\max}=0.0749.$$

Since equations (8.10) and (8.14) have no root in common, it would be noticed that an inequality

(8.16) $$(I_{ABO}-J_{ABO})^{\max}>(I_{ABO})^{\max}-(J_{ABO})^{\max}$$

must hold in the strict sense, although both sides differ so slightly that an actual difference is yet invisible at the beginning four decimal places. (Cf. (7.28) of VII; in fact, $I_{ABO}\equiv P_{ABO}$.)

The whole probability against at least one child, given by

$$(8.17)\quad \tilde{J}_{ABO}=2I_{ABO}-J_{ABO} =\tfrac{1}{2}p(3-p)(1-p)^4+\tfrac{1}{2}q(3-q)(1-q)^4+\tfrac{1}{4}pqr^2(20-9r-7r^2),$$

will be maximized by the distribution

$$(8.18)\quad \begin{array}{lll} p=q=0.2210, & r=0.5580; & \\ \bar{O}=0.3113, & \bar{A}=\bar{B}=0.2955, & \overline{AB}=0.0977; \end{array}$$

p and q being a root of the quintic equation

$$(8.19)\quad 348x^5-810x^4+536x^3-177x^2+48x-6=0.$$

The maximum of $\tilde{J}_{ABO}$ is equal to

$$(8.20)\quad (\tilde{J}_{ABO})^{\max}=0.2748.$$

We notice, by the way, that, more precisely calculated, really an inequality must hold:

$$(8.21)\quad (\tilde{J}_{ABO})^{\max}\equiv(2I_{ABO}-J_{ABO})^{\max}<(I_{ABO}-J_{ABO})^{\max}+(I_{ABO})^{\max}.$$

We next consider Q blood type. In a similar way as above, the following results will be obtained.

$$(8.22)\quad J_Q=\tfrac{1}{2}u(1+u)v^4;$$

maximizing distribution and maximum value:

$$(8.23)\quad \begin{array}{ll} u=(\sqrt{33}-3)/12=0.2287, & v=(15-\sqrt{33})/12=0.7713; \\ \bar{Q}=(15\sqrt{33}-57)/72=0.4051, & \bar{q}=(129-15\sqrt{33})/72=0.5949; \end{array}$$

$$(8.24)\quad (J_Q)^{\max}=(561\sqrt{33}-2879)/2^8 3^3=0.0497.$$

$$(8.25)\quad I_Q-J_Q=\tfrac{1}{2}uv^5;$$

maximizing distribution and maximum value:

$$(8.26)\quad u=1/6,\quad v=5/6;\quad \bar{Q}=11/36,\quad \bar{q}=25/36;$$

$$(8.27)\quad (I_Q-J_Q)^{\max}=5^5/2\cdot 6^6=0.0335.$$

$$(8.28)\quad \tilde{J}_Q=\tfrac{1}{2}uv^4(2+v);$$

maximizing distribution and maximum value:

$$(8.29)\quad \begin{array}{ll} u=(17-\sqrt{217})/12=0.1891, & v=(\sqrt{217}-5)/12=0.8109; \\ \bar{Q}=(5\sqrt{217}-49)/72=0.3424, & \bar{q}=(121-5\sqrt{217})/72=0.6576; \end{array}$$

$$(8.30)\quad (\tilde{J}_Q)^{\max}=(331517-21049\sqrt{217})/2^8\cdot 3^3=0.1149.$$

Inequalities analogous to (8.16) and (8.21) may be noticed.

If the results on Q blood type are compared with those on MN blood type, the probabilities in the former are less than the corresponding ones in the latter. The deficiencies are evidently caused by the existence of a recessive gene.

In conclusion, we consider the general inherited character with genes $A_i\,(1\leqq i\leqq m)$. The values of whole probabilities being expressions symmetric with respect to $p_i\,(1\leqq i\leqq m)$, they will be stationary for the symmetric distribution

$$p_i=1/m \qquad (i=1,\ldots,m). \tag{8.31}$$

For this distribution, we get

$$(J)^{\text{stat}}=\frac{1}{4}\Big(1-\frac{1}{m}\Big)\Big(4-\frac{8}{m}-\frac{8}{m^2}+\frac{39}{m^3}-\frac{31}{m^4}\Big), \tag{8.32}$$

$$(I-J)^{\text{stat}}=\frac{1}{4m}\Big(1-\frac{1}{m}\Big)\Big(4-\frac{27}{m^2}+\frac{31}{m^3}\Big), \tag{8.33}$$

$$(\tilde{J})^{\text{stat}}=\frac{1}{4}\Big(1-\frac{1}{m}\Big)\Big(4-\frac{8}{m^2}-\frac{15}{m^3}+\frac{31}{m^4}\Big). \tag{8.34}$$

If we differentiate the quantities (8.32), (8.33) and (8.34) with respect to $1/m$, regarded as if a continuous variable, we obtain

$$\frac{d}{d(1/m)}(J)^{\text{stat}}=-\frac{1}{4}\Big(1-\frac{2}{m}\Big)\Big(12\Big(1-\frac{2}{m}\Big)\Big(1+\frac{4}{m}\Big)+\frac{3}{m^2}+\frac{94}{m^3}\Big)-\frac{33}{4m^4}, \tag{8.35}$$

$$\frac{d}{d(1/m)}(I-J)^{\text{stat}}=\frac{1}{4}\Big(1-\frac{5}{m}\Big)\Big(4+\frac{12}{m}-\frac{21}{m^2}+\frac{127}{m^3}\Big)+\frac{120}{m^4}, \tag{8.36}$$

$$\frac{d}{d(1/m)}(\tilde{J})^{\text{stat}}=-\frac{1}{4}\Big(1-\frac{2}{m}\Big)\Big(4+\frac{24}{m}+\frac{69}{m^2}-\frac{46}{m^3}\Big)-\frac{63}{4m^4}. \tag{8.37}$$

Thus, (8.35) and (8.37) remain negative for $m\geqq 2$. Hence, the quantities (8.32) and (8.34) increase as m increases, and they tend asymptotically to the common limit 1 as $m\to\infty$. But, (8.36) remains positive for $m\geqq 5$ ——— while it is negative for $m\leqq 4$ ———. Hence, the quantity (8.33) decreases as m increases provided $m\geqq 5$ and tends to the limit 0 as $m\to\infty$.

By the way, the inequalities

$$J\leqq I\leqq \tilde{J} \tag{8.38}$$

are evident, in view of the meanings of the quantities involved.