53. Probability-theoretic Investigations on Inheritance. IX$_3$. Non-Paternity Concerning Mother-Children Combinations.

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5. Decomposition of the probability J with regard to types of children.

We now proceed to decompose the whole probability J in (4.25) into sub-probabilities with respect to pairs of children types. Corresponding to (3.1), we denote by

$$K(hk, fg) = \sum_{i \leq j} Q(ij; hk, fg) \tag{5.1}$$

the sub-probability of proving non-paternity against both children (A_{hk}, A_{fg}).

In order to calculate the value of (5.1), it will again be convenient to consider an excess of (3.1). In view of (4.6), an inequality

$$K(hk, fg) \leqq H(hk, fg) \tag{5.2}$$

holds in general, while, in particular, a useful equality

$$K(fg, fg) = H(fg, fg) \tag{5.3}$$

holds good. The results corresponding to (3.2) to (3.10) are as follows:

$$K(ff, ff) = H(ff, ff), \tag{5.4}$$

$$K(hh, ff) = H(hh, ff) - \tfrac{1}{4}p_f^2 p_h^3 (2 - 2p_f - p_h) \qquad (h \neq f), \tag{5.5}$$

$$K(hf, ff) = H(hf, ff) - \tfrac{1}{2}p_f^2 p_h^2 (1 + p_f)(2 - 2p_f - p_h) \qquad (h \neq f), \tag{5.6}$$

$$K(hk, ff) = H(hk, ff) - \tfrac{1}{4}p_f^2 p_h p_k (2(1 - p_f)(p_h + p_k) - (p_h^2 + p_k^2)) \qquad (h, k \neq f;\ h \neq k); \tag{5.7}$$

$$K(ff, fg) = H(ff, fg) - \tfrac{1}{4}p_f^3 p_g (2 + p_f - p_f^2 - (4 + p_f)p_g + 2p_g^2) \qquad (f \neq g), \tag{5.8}$$

$$K(fg, fg) = H(fg, fg), \tag{5.9}$$

$$\begin{aligned} K(hf, fg) = H(hf, fg) &- \tfrac{1}{4}p_f p_g p_h (p_f^2 (2 - p_f - 2p_g) \\ &+ (2 + 8p_f - 2p_g - 5p_f^2 - 10 p_f p_g) p_h - (1 + 5p_f) p_h^2) \\ &(f \neq g;\ h \neq f, g), \end{aligned} \tag{5.10}$$

$$K(hh, fg) = H(hh, fg) - \tfrac{1}{2}p_f p_g p_h^3 (2 - (p_f + p_g) - p_h) \qquad (f \neq g;\ h \neq f, g), \tag{5.11}$$

$$\begin{aligned} K(hk, fg) = H(hk, fg) & \\ - \tfrac{1}{2}p_f p_g p_h p_k & ((2 - p_f - p_g)(p_h + p_k) - (p_h^2 + p_k^2)) \\ & (f \neq g;\ h, k \neq f, g;\ h \neq k). \end{aligned} \tag{5.12}$$

We can further derive the results corresponding to (3.11) to (3.16), stating as follows:

(5.13) $$\sum_{h\neq f} K(hh, ff) = \sum_{h\neq f} H(hh, ff) - \tfrac{1}{4}p_f^2(2S_3 - S_4 - 2S_3p_f - 2p_f^3 + 3p_f^4),$$

(5.14) $$\sum_{h\neq f} K(hf, ff) = \sum_{h\neq f} H(hf, ff) - \tfrac{1}{2}p_f^2(2S_2 - S_3 - S_3p_f - 2(1+S_2)p_f^2 + p_f^3 + 3p_f^4),$$

(5.15) $$\sum_{h,\,k\neq f}{}' K(hk, ff) = \sum_{h,\,k\neq f}{}' H(hk, ff) - \tfrac{1}{4}p_f^2(2S_2 - 3S_3 + S_4 - (4S_2 - 3S_2)p_f - 2(1-S_2)p_f^2 + 7p_f^3 - 6p_f^4);$$

(5 16) $$\sum_{h\neq f,\,g} (K(hf, fg) + K(hg, fg)) = \sum_{h\neq f,\,g} (H(hf, fg) + H(hg, fg)) - \tfrac{1}{4}p_fp_g(2(2S_2 - S_3) + (6S_2 - 5S_3)(p_f + p_g) - (2+5S_2)(p_f^2 + p_g^2) - 20S_2p_fp_g - 7(p_f^3 + p_g^3) - 10p_fp_g(p_f + p_g) + 11(p_f^4 + p_g^4) + 28p_fp_g(p_f^2 + p_g^2) + 14p_f^2p_g^2) \qquad (f\neq g),$$

(5.17) $$\sum_{h\neq f,\,g} K(hh, fg) = \sum_{h\neq f,\,g} H(hh, fg) - \tfrac{1}{2}p_fp_g(2S_3 - S_4 - S_3(p_f + p_g) - 2(p_f^3 + p_g^3) + 2(p_f^4 + p_g^4) + p_fp_g(p_f^2 + p_g^2)) \qquad (f\neq g),$$

(5.18) $$\sum_{h,k\neq f,g}{}' K(hk, fg) = \sum_{h,k\neq f,g}{}' H(hk, fg) - \tfrac{1}{2}p_fp_g(2S_2 - 3S_3 + S_4 - 3S_2(p_f + p_g) - (2 - S_2)(p_f^2 + p_g^2) + 2S_2p_fp_g + 6(p_f^3 + p_g^3) + 3p_fp_g(p_f + p_g) - 4(p_f^4 + p_g^4) - 4p_fp_g(p_f^2 + p_g^2) - 2p_f^2p_g^2) \qquad (f\neq g).$$

Although, in (5.13) to (5.18), the classification is based upon the types of second child, it may be noticed that the results classified by the types of first child is also simultaneously obtained. In fact, the symmetry character of Q represented by (4.4) implies that of K; namely,

(5.19) $$K(hk, fg) = K(fg, hk).$$

Next, we consider, corresponding to (3.17) and (3.18), the probability of proving non-paternity against both children separately among which the one, without loss of generality the second say, is of a fixed homozygote A_{ff} or heterozygote A_{fg} $(f\neq g)$. We then get

(5.20) $$T(ff) \equiv K(ff, ff) + \sum_{h\neq f} (K(hh, ff) + K(hf, ff)) + \sum_{h,\,k\neq f}{}' K(hk, ff) = R(ff) - \tfrac{1}{4}p_f^2(3(2S_2 - S_3) - (4S_2 + S_3)p_f - 2(3 + S_2)p_f^2 + 7p_f^3 + 3p_f^4);$$

(5.21) $$T(fg) \equiv K(ff, fg) + K(gg, fg) + K(fg, fg) + \sum_{h\neq f,\,g} (K(hf, fg) + K(hg, fg) + K(hh, fg)) + \sum_{h,\,k\neq f,\,g}{}' K(hk, fg) = R(fg) - \tfrac{1}{4}p_fp_g(4(2S_2 - S_3) - 7S_3(p_f + p_g) - (4 + 3S_2)(p_f^2 + p_g^2) - 16S_2p_fp_g + 3(p_f^3 + p_g^3) - 8p_fp_g(p_f + p_g) + 6(p_f^4 + p_g^4) + 21p_fp_g(p_f^2 + p_g^2) + 14p_f^2p_g^2) \qquad (f\neq g).$$

Summing up the last two expressions over all the possible suffices, the whole probability (4.25) will again be obtained; namely,

$$(5.22)\qquad J=\sum_{f=1}^{m} T(ff)+\sum_{f,g}{}' T(fg).$$

But, we now shall, more precisely, calculate the sub-probabilities in the following forms:

$$(5.23)\qquad \sum_{f=1}^{m} K(ff,ff)=\tfrac{1}{4}S_2-\tfrac{1}{2}S_4+\tfrac{1}{4}S_6,$$

$$(5.24)\qquad \sum_{f=1}^{m}\sum_{h\neq f} K(hh,ff)=\tfrac{1}{4}S_2^2-\tfrac{1}{4}S_4-S_2S_3+S_5+\tfrac{1}{2}S_3^2+\tfrac{1}{2}S_2S_4-S_6,$$

$$(5.25)\qquad \sum_{f=1}^{m}\sum_{h\neq f} K(hf,ff)=\tfrac{1}{2}S_2-S_3-S_2^2+S_4+\tfrac{1}{2}S_2S_3+\tfrac{1}{2}S_5+\tfrac{1}{2}S_3^2+S_2S_4-2S_6,$$

$$(5.26)\qquad \sum_{f=1}^{m}\sum_{h,k\neq f}{}' K(hk,ff)$$
$$=\tfrac{1}{4}S_2-S_3-\tfrac{3}{4}S_2^2+\tfrac{9}{4}S_4+\tfrac{9}{4}S_2S_3-\tfrac{13}{4}S_5-\tfrac{3}{4}S_3^2-S_2S_4+2S_6;$$

$$(5.27)\qquad \sum_{f=1}^{m}\sum_{g\neq f} K(ff,fg)=\tfrac{1}{2}S_2-S_3-S_2^2+S_4+\tfrac{1}{2}S_2S_3+\tfrac{1}{2}S_5+\tfrac{1}{2}S_3^2+S_2S_4-2S_6,$$

$$(5.28)\qquad \sum_{f,g}{}' K(fg,fg)$$
$$=\tfrac{1}{4}-\tfrac{1}{4}S_2-\tfrac{1}{4}S_3-\tfrac{1}{2}S_2^2+S_4-S_2S_3+\tfrac{3}{4}S_5+\tfrac{3}{4}S_3^2+\tfrac{5}{4}S_2S_4-2S_6,$$

$$(5.29)\qquad \sum_{f,g}{}'\sum_{h\neq f,g}(K(hf,fg)+K(hg,fg))$$
$$=\tfrac{1}{2}-S_2+\tfrac{3}{4}S_3-3S_2^2-\tfrac{19}{4}S_4+\tfrac{51}{4}S_2S_3-15S_5+\tfrac{5}{2}S_2^3-5S_3^2-\tfrac{53}{4}S_2S_4+\tfrac{33}{4}S_6,$$

$$(5.30)\qquad \sum_{f,g}{}'\sum_{h\neq f,g} K(hh,fg)$$
$$=\tfrac{1}{4}S_2-S_3-\tfrac{3}{4}S_2^2+\tfrac{9}{4}S_4+\tfrac{9}{4}S_2S_3-\tfrac{13}{4}S_5-\tfrac{3}{4}S_3^2-S_2S_4+2S_6,$$

$$(5.31)\qquad \sum_{f,g}{}'\sum_{h,k\neq f,g}{}' K(hk,fg)=\tfrac{1}{4}-\tfrac{5}{2}S_2+\tfrac{11}{2}S_3+\tfrac{17}{4}S_2^2-\tfrac{19}{2}S_4-\tfrac{17}{4}S_2S_3+11S_5$$
$$-\tfrac{1}{2}S_2^3+2S_3^2+4S_2S_4-6S_6.$$

The sum of (5.23) to (5.26) yields the first sum of the right-hand side in (5.22):

$$(5.32)\qquad \sum_{f=1}^{m} T(ff)=S_2-2S_3-\tfrac{3}{2}S_2^2+\tfrac{5}{2}S_4+\tfrac{7}{4}S_2S_3-\tfrac{7}{4}S_5+\tfrac{1}{4}S_3^2+\tfrac{1}{2}S_2S_4-\tfrac{3}{4}S_6,$$

while the sum of (5.27) to (5.31) yields the second sum:

$$(5.33)\qquad \sum_{f,g}{}' T(fg)=1-4S_2+\tfrac{9}{2}S_3-S_2^2-\tfrac{1}{2}S_4$$
$$+6S_2S_3-6S_5+2S_2^3-\tfrac{5}{2}S_3^2-8S_2S_4+\tfrac{17}{2}S_6.$$

That the sum of (5.32) and (5.33) implies the whole probability J in (4.25) is the matter of course.

The case of mixed mother-children can correspondingly be discussed, and the whole probability in (4.31) will be obtained also in such a procedure.

6. Non-paternity against a distinguished child alone.

We have discussed the problems to determine the probability of proving non-paternity against a distinguished child at any rate and then to determine that against both children separately; in either case, a mother-children combination with two children being given. Now, we can easily determine the probability of proving non-paternity against a distinguished child alone, i.e., the probability of the event that non-paternity proof is possible against a distinguished child but impossible (of course, not affirmative!) against another child.

Let a distinguished child be, without loss of generality, the second child. If the case deniable against both children separately is excluded from the case deniable against second child at any rate, then the case deniable against second child alone will remain. Hence, the fundamental quantity for the present problem is represented as the difference between (2.3) and (4.2). That is, the difference

$$(6.1)\quad P(ij;hk,fg)-Q(ij;hk,fg)=\pi(ij;hk,fg)(V(ij;fg)-V(ij;hk,fg)).$$

represents *the probability of proving non-paternity against second child A_{fg} alone* accompanied by its mother A_{ij} and her first child A_{hk}; non-paternity proof against first child being assumed to be impossible. Correspondingly, there appear differences such as

$$(6.2)\quad I(ij;hk)-J(ij;hk),\qquad H(hk,fg)-K(hk,fg),\qquad \text{etc.}$$

In §§ 4 and 5, we have frequently considered such differences for the convenience in calculation. Now, those may be regarded as the probabilities in the present problem. In particular, the *whole probability* of proving non-paternity against a distinguished child is, in view of (2.17) and (4.25), given by

$$(6.3)\quad I-J=S_2-\tfrac{3}{2}S_3+\tfrac{1}{2}S_2^2-\tfrac{19}{4}S_2S_3+\tfrac{19}{4}S_5-2S_2^3+\tfrac{9}{4}S_3^2+\tfrac{15}{2}S_2S_4-\tfrac{31}{4}S_6.$$

The whole probability in mixed case is, in view of (4.12) in VII and (4.31), given by

$$(6.4)\quad \begin{aligned} I'-J'=&S_2'-\tfrac{3}{2}S_3'+\tfrac{1}{2}(-2S_2'^2+3S_{1,1}^2)\\ &+\tfrac{3}{2}(S_4'-S_{2,2})-\tfrac{1}{4}(-2S_2'S_{1,2}+19S_{1,1}S_{1,2}+2S_2'S_{2,1})+\tfrac{19}{4}S_{2,3}\\ &-2S_2'^2S_2+\tfrac{9}{4}S_{1,2}^2+\tfrac{1}{2}(11S_{1,1}S_{1,3}+4S_2'S_{2,2})-\tfrac{31}{4}S_{2,4}, \end{aligned}$$

an evident identity $I'=P'$ being to be remembered.

7. Probability against at least one child.

We have discussed in § 4 the problem of proving non-paternity against both children of the same family *separately*, i.e., against both first and second children *indifferent to* second and first children respectively. There will, however, arise a more proper problem in case of non-paternity proof against *both children of the same family.* Namely, if it is sure that both children of the same family are presented, the non-paternity proof would be established against both children provided that it is established against at least one child among them. In other words, a man not compatible with at least one of children of the same family could then simultaneously assert his non-paternity also against another child and hence against both children. We shall now discuss such a problem.

Let now a mother-children combination belonging to the same $(A_{ij}; A_{hk}, A_{fg})$ be given. We introduce, instead of (4.1), a quantity family

$$\tilde{V}(ij; hk, fg), \tag{7.1}$$

which represents the probability of proving non-paternity of a man chosen at random against at least one child, a fact that both children belong to the same family being taken into account. As stated above, the non-paternity proof against both children would then simultaneously be established. By remembering also the combination-probability, *the probability of proving non-paternity against at least one child* (*and hence simultaneously against both children*) becomes then

$$\tilde{Q}(ij; hk, fg) = \pi(ij; hk, fg)\tilde{V}(ij; hk, fg). \tag{7.2}$$

Now, if, from the cases deniable against first child at any rate and against second at any rate, the case against both children being repeatedly taken into account, is excluded once, then there remains the case deniable against at least one child of the same family and hence against both children. Consequently, we get a fundamental interrelation

$$\tilde{V}(ij; hk, fg) = V(ij; fg) + V(ij; hk) - V(ij; hk, fg), \tag{7.3}$$

whence it follows, by multiplying by $\pi(ij; hk, fg)$,

$$\tilde{Q}(ij; hk, fg) = P(ij; hk, fg) + P(ij; fg, hk) - Q(ij; hk, fg). \tag{7.4}$$

Thus, the quantity $\tilde{Q}(ij; hk, fg)$ may also be regarded as the sum of the probability of proving non-paternity against second child at any rate and that against first child alone.

Summing up the quantities in (7.2) over all possible sets of indices, we get the *whole probability* $\tilde{J}$ of proving non-paternity against at least one child of the same family, and hence against both children. In view of (7.4), we get immediately the result, stating

$$\tilde{J} = 2I - J = 1 - S_2 - \tfrac{1}{2}S_3 - \tfrac{3}{2}S_2^2 + 2S_4 - \tfrac{7}{4}S_2S_3 + \tfrac{7}{4}S_5 - 2S_2^3 + \tfrac{9}{4}S_3^2 + \tfrac{15}{2}S_2S_4 - \tfrac{31}{4}S_6. \tag{7.5}$$

A symmetric character of the $\tilde{V}$'s and hence that of the $\tilde{Q}$'s, corresponding to (4.3) to (4.4), is obvious. From definition, the relations corresponding to (4.5) to (4.8) are also obvious, i.e.,

$$V(ij;\, hk, fg) \leqq V(ij;\, hk) \leqq \tilde{V}(ij;\, hk, fg), \tag{7.6}$$

$$Q(ij;\, hk, fg) \leqq P(ij;\, hk) \leqq \tilde{Q}(ij;\, hk, fg); \tag{7.7}$$

$$V(ij;\, fg, fg) = V(ij;\, fg) = \tilde{V}(ij;\, fg, fg), \tag{7.8}$$

$$Q(ij;\, fg, fg) = P(ij;\, fg) = \tilde{Q}(ij;\, fg, fg). \tag{7.9}$$

The relation (7.3), or rather (7.4), having been established, the partial sums, corresponding to (2.6) to (2.10) or (4.13) to (4.17), will easily be written down. Since the present problem is a proper one with respect to the same family, we shall state the results explicitly in the following lines.

— To be continued —