59. Probability-theoretic Investigations on Inheritance. X_1. Non-Paternity Concerning Mother-Child-Child Combinations.

By Yûsaku KOMATU.

Department of Mathematics, Tokyo Institute of Technology and
Department of Legal Medicine, Tokyo Medical and Dental University.

(Comm. by T. FURUHATA, M.J.A., May 13, 1952.)

1. Non-paternity against a distinguished child.

Problems discussed in the preceding chapter[1] have exclusively concerned two children belonging to the same family, that is, possessing a father also in common. There arise analogous problems concerning two children possessing a mother alone in common, which will be discussed in the present chapter. While the former problems have depended on *mother-children combinations*, the latter ones depend on *mother-child-child combinations*.

Now, consider a triple consisting of a mother A_{ij}, her first child A_{hk} and her second child A_{fg}, both children being assumed not to possess a common father. The probability of an event that such a triple appears and then a man chosen at random can assert his *non-paternity against second child at any rate* is, corresponding to a former expression (2.3) of IX, represented by

$$(1.1) \qquad P_0(ij; hk, fg) \equiv \pi_0(ij; hk, fg) V(ij; fg);$$

the π_0's denoting the probabilities of mother-child-child combination defined in (5.9) of IV and V's the quantities introduced in (2.1) of VII. This is a basic quantity and can, in view of (5.7) of IV and (2.2) of VII, i.e.,

$$\pi_0(ij; hk, fg) = \pi(ij; hk)\pi(ij; fg)/\bar{A}_{ij}, \quad \pi(ij; fg) V(ij; fg) = P(ij; fg),$$

be expressed also in the form

$$(1.2) \qquad P_0(ij; hk, fg) = P(ij; fg) \cdot \pi(ij; hk)/\bar{A}_{ij};$$

1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethren combinations; VI. Rate of danger in random blood transfusion; VII. Non-paternity problems; VIII. Further discussions on non-paternity problems; IX. Non-paternity concerning mother-children combinations. Proc. Japan Acad. **27** (1951), I. 371-377; II. 378-383, 384-387; III. 459-464, 466-471, 472-477, 478-483; IV. 587-592, 596-597, 598-603, 605-610, 611-614, 615-620; V. 689-693, 694-699; **28** (1952), VI. 54-58; VII. 102-104, 105-108, 109-111, 112-115, 116-120, 121-125; VIII. 162-164, 165-168, 169-171; IX. 207-212, 213-217, 218-223, 224-229. These papers will be referred to as I; II; III; IV; V; VI; VII; VIII; IX.

the coefficient of $P(ij;fg)$ in the right-hand side being nothing but the quantity noticed in (1.27) of IV, the fact which also implies immediately the relation (1.2) itself.

The coefficient in consideration being independent of the type of second child, the partial sum

$$(1.3) \qquad I_0(ij;hk)=\sum_{f\leqq g} P_0(ij;hk,fg),$$

which denotes the quantity corresponding to (2.5) of VII in the preceding case, can easily be determined. In fact, making use of the notation introduced in (2.3) of VII, we get an expression

$$(1.4) \qquad I_0(ij;hk)=P(ij)\pi(ij;hk)/\bar{A}_{ij}.$$

If we sum up the last relations over all the possible pairs of suffices h and k, we then obtain

$$(1.5) \qquad \sum_{h\leqq k} I_0(ij;hk)=P(ij);$$

here use being made of an evident relation (1.20) of IV, i.e., $\sum_{h\leqq k}\pi(ij;hk)=\bar{A}_{ij}$. If we further sum up the relations (1.5) over i and j, the whole probability (2.20) of VII will be reproduced (cf. also (2.19) of VIII and L in (4.33) below), namely,

$$(1.6) \quad I_0\equiv\sum_{i\leqq j;\, h\leqq k} I_0(ij;hk)=1-2S_2+S_3-2S_2^2+2S_4+3S_2S_3-3S_5=P.$$

Generalization of the above discussion to the mixed case is also immediate. We shall assume here that a putative father belongs to the same population as a true father. In general, given a quantity X dependent on $\{p_i\}$, let the quantity obtained from X by replacing all the $p_i(i=1,\ldots,m)$ by the corresponding p_i'' be denoted by $[X]''$. Then, the quantity

$$(1.7) \qquad V''(ij;fg)\equiv[V(ij;fg)]''$$

represents the probability of an event that, given a second child A_{fg} produced by a mother A_{ij} belonging to a poplulation with distribution $\{p_i\}$ and by a father belonging to a population with distribution $\{p_i''\}$, a man belonging to the same population as a true father can assert his non-paternity. Thus we now get, instead of (1.1), a basic quantity

$$(1.8) \qquad P^*(ij;hk,fg)=\pi^*(ij;hk,fg)\; V''(ij;fg),$$

the π^*'s being introduced in (5.6) of IV; here a father of first child is assumed to belong to a population with distribution $\{p_i'\}$. Hence, we get, corresponding to (1.2),

$$(1.9) \quad P^*(ij;hk,fg)=P''(ij;fg)\pi'(ij;hk)/\bar{A}_{ij}=\pi'(ij;hk)\,[P(ij;fg)/\bar{A}_{ij}]''.$$

Thus, the relations corresponding to (1.3), (1.4); (1.5), (1.6) will become

$$(1.10) \qquad I^*(ij;hk)\equiv\sum_{f\leqq g} P(ij;hk,fg)=P''(ij)\pi'(ij;hk)/A_{ij};$$

$$(1.11) \quad \sum_{h\leqq k} I^*(ij;hk)=P''(ij)\equiv\sum_{f\leqq h} P''(ij;fg)=p_ip_j[P(ij)/p_ip_j]'',$$

$$(1.12) \qquad I^*=P'',$$

respectively, P'' denoting the expression obtained from (4.12) of VII by replacing the p''s by the corresponding p'''s.

2. Non-paternity against both children.

In discussion of the preceding section, it has been quite indifferent whether a man putative against second child is or is not a true father of the first child. We now consider a problem of determining the probability of proving non-paternity against both children of a mother-child-child combination. Given such a combination $(A_{ij}; A_{hk}, A_{fg})$, the probability of an event that a man chosen at random can assert his *non-paternity against both children* is represented by

$$Q_0(ij; hk, fg) \equiv \pi_0(ij; hk, fg)\, V(ij; hk, fg); \tag{2.1}$$

the π_0's and the V's being of the same meaning as in (5.9) of IV and (4.1) of IX, respectively. The partial sum

$$J_0(ij; hk) = \sum_{f \leq g} Q_0(ij; hk, fg), \tag{2.2}$$

corresponding to (1.3), can be represented, in view of (5.7) of IV, also in the form

$$J_0(ij; hk) = \frac{\pi(ij; hk)}{A_{ij}} \sum_{f \leq g} \pi(ij; fg)\, V(ij; hk, fg). \tag{2.3}$$

The values of (2.2) can, in separate cases, explicitly determined as follows:

$$\begin{aligned} J_0(ii;\, ii) &= Q_0(ii; ii, ii) + \sum_{k \neq i} Q_0(ii; ii, ik) \\ &= p_i^3(1 - 2S_2 + S_3 - 2(1-S_2)p_i + 3p_i^2 - 3p_i^3), \end{aligned} \tag{2.4}$$

$$\begin{aligned} J_0(ii; ih) &= Q_0(ii; ih, ii) + Q_0(ii; ih, ih) + \sum_{k \neq i, h} Q_0(ii; ih, ik) \\ &= p_i^2 p_h(1 - 2S_2 + S_3 - 2(1-S_2)p_h + 3p_h^2 - 3p_h^3) \qquad (h \neq i); \end{aligned} \tag{2.5}$$

$$\begin{aligned} J_0(ij,\, ii) &= Q_0(ij; ii, ii) + Q_0(ij; ii, jj) + Q_0(ij; ii, ij) \\ &\quad + \sum_{k \neq i, j} (Q_0(ij; ii, ik) + Q_0(ij; ii, jk)) \\ &= p_i^2 p_j(1 - 2S_2 + S_3 - 2(1-S_2)p_i + 3p_i^2 - p_i p_j \\ &\quad - 3p_i^3 + p_i p_j(p_i + \tfrac{1}{2}p_j)) \qquad (i \neq j), \end{aligned} \tag{2.6}$$

$$\begin{aligned} J_0(ij;\, ij) &= Q_0(ij; ij, ii) + Q_0(ij; ij, jj) + Q_0(ij; ij, ij) \\ &\quad + \sum_{k \neq i, j} (Q_0(ij; ij, ik) + Q_0(ij; ij, jk)) \\ &= p_i p_j(p_i + p_j)(1 - 2S_2 + S_3 - 2(1-S_2)(p_i + p_j) + 3(p_i^2 + p_j^2) \\ &\quad + 2p_i p_j - 3(p_i^3 + p_j^3) - 2p_i p_j(p_i + p_j)) \qquad (i \neq j), \end{aligned} \tag{2.7}$$

$$\begin{aligned} J_0(ij; ih) &= Q_0(ij; ih, ii) + Q_0(ij; ih, jj) + Q_0(ij; ih, ij) + Q_0(ij; ih; ih) \\ &\quad + Q_0(ij; ih, jh) + \sum_{k \neq i, j, h} (Q_0(ij; ih, ik) + Q_0(ij, ih, jk)) \\ &= p_i p_j p_h(1 - 2S_2 + S_3 - 2p_i p_j + \tfrac{3}{2}p_i p_j(p_i + p_j) \\ &\quad - 2(1 - S_2 - p_i p_j)p_h + 3p_h^2 - 3p_h^3) \qquad (i \neq j;\, h \neq i, j). \end{aligned} \tag{2.8}$$

These relations (2.4) to (2.8) correspond to (4.13) to (4.17) of IX, respectively. Corresponding to (4.18) to (4.22) of IX, we get the following results:

$$(2.9)\qquad \sum_{i=1}^{m} J_0(ii;ii)=S_3-2S_4-2S_2S_3+3S_5+S_3^2+2S_2S_4-3S_6,$$

$$(2.10)\qquad \sum_{i=1}^{m}\sum_{h\neq i} J_0(ii;ih)=S_2-S_3-4S_2^2+2S_4+6S_2S_3 -3S_5+2S_2^3-S_3^2-5S_2S_4+3S_6;$$

$$(2.11)\qquad \sum_{i,j}{}' (J_0(ij;ii)+J_0(ij;jj))=S_2-3S_3-2S_2^2+5S_4 +4S_2S_3-5S_5-\tfrac{1}{2}S_3^2-S_2S_4+\tfrac{3}{2}S_6,$$

$$(2.12)\qquad \sum_{i,j}{}' J_0(ij;ij)=S_2-3S_3-4S_2^2+7S_4+10S_2S_3-11S_5 +2S_2^3-3S_3^2-9S_2S_4+10S_6,$$

$$(2.13)\qquad \sum_{i,j}{}'\sum_{h\neq i,j}(J_0(ij;ih)+J_0(ij;jh))=1-7S_2+10S_3+8S_2^2-11S_4 -7S_2S_3+5S_5-S_3^2-2S_2S_4+4S_6.$$

The sum of (2.9) and (2.10) yields the partial sum of probabilities of proving non-paternity against both children of a mother-child-child combination over homozygotic mothers:

$$(2.14)\qquad S_2(1-4S_2+4S_3+2S_2^2-3S_4),$$

while the sum of (2.11) to (2.13) yields that over heterozygotic mothers:

$$(2.15)\qquad 1-5S_2+4S_3+2S_2^2+S_4+7S_2S_3-11S_5+2S_2^3-\tfrac{9}{2}S_3^2-12S_2S_4+\tfrac{31}{2}S_6.$$

The sum of the last two expressions (2.14) and (2.15) represents the *whole probability of proving non-paternity against both children of a mother-child-child combination*, stating

$$(2.16)\qquad J_0=1-4S_2+4S_3-2S_2^2+S_4+11S_2S_3-11S_5 +4S_2^3-\tfrac{9}{2}S_3^2-15S_2S_4+\tfrac{31}{2}S_6.$$

In case of *MN* blood type which may be regarded as a special case of general development, the whole probability becomes briefly

$$(2.17)\qquad J_{0MN}=s^2t^2(2-3st).$$

The case where recessive genes are existent and phenotypes alone are available for judgement can also, in principle, quite similarly be treated. For instance, we obtain the following expressions for *whole probabilities in various blood types*:

$$(2.18)\qquad J_{0ABO}=p^2(1-p)^4+q^2(1-q)^4+\tfrac{1}{2}pqr^2(1+7r^2),$$

$$(2.19)\qquad J_{0Q}=u^2v^4,$$

$$(2.20)\qquad J_{0Qq\pm}=u^2v^4+(2u+v_1)v_1v_2^4.$$

The results obtained hitherto in the present section can be generalized to mixed case. But, the detail will be left to the reader.

By comparing the probabilities J_0's in (2.18), (2.19), (2.20) with the corresponding probabilities J's in (4.26), (4.27), (4.28) of IX respectively, we know that *the inequalities*

$$(2.21)\qquad J_{0ABO}\leqq J_{ABO},\quad J_{0Q}\leqq J_{Q},\quad J_{0Qq\pm}\leqq J_{Qq\pm}$$

hold good, with inequality sign except trivial distributions. In fact, we can verify (2.21) as follows:

$$\begin{aligned}J_{ABO}-J_{0ABO}&=\tfrac{1}{2}p(1-p)^5+\tfrac{1}{2}q(1-q)^5+\tfrac{1}{4}pqr^2(2+r-7r^2)\\&=\tfrac{1}{2}p(q+r)^5+\tfrac{1}{2}q(p+r)^5+\tfrac{1}{4}pqr^2(2+r-7r^2)\\&\geqq\tfrac{1}{2}p\cdot 5qr^4+\tfrac{1}{2}q\cdot 5pr^4+\tfrac{1}{4}pqr^2(2+r-7r^2)\\&=\tfrac{1}{4}pqr^2(2+r+13r^2)\geqq 0,\end{aligned}$$

$$J_Q-J_{0Q}=\tfrac{1}{2}uv^5\geqq 0,\qquad J_{Qq\pm}-J_{0Qq\pm}=\tfrac{1}{2}(u(v^5-v_1v_2^4)+v_1v_2^5)\geqq 0.$$

Comparison of (2.17) with (8.1) of IX shows

$$(2.22)\qquad J_{0MN}\leqq J_{MN};$$

equality sign being exclusive unless $st=0$. In fact, remembering $0\leqq st\leqq 1/4$, we have

$$J_{MN}-J_{0MN}=\tfrac{1}{4}st(1-2st)(2-3st)\geqq 0.$$

An analogous inequality between (2.16) and (4.25) of IX, i.e.,

$$(2.23)\qquad J_0\leqq J$$

does hold also good. We can conclude moreover that the corresponding inequalities between (2.14) and (4.23) of IX and between (2.15) and (4.24) of IX are also valid. For instance, the difference of (2.14) and (4.23) of IX becomes, in fact,

$$\begin{aligned}&S_2(1-3S_2+\tfrac{5}{2}S_3+S_2^2-\tfrac{3}{2}S_4)-S_2(1-4S_2+4S_3+2S_2^2-3S_4)\\&=S_2(S_2-\tfrac{3}{2}S_3-S_2^2+\tfrac{3}{2}S_4)=S_2\sum_{i=1}^{m}\sum_{h\neq i}p_i^2p_h(1-\tfrac{1}{2}p_i-p_h)\geqq 0.\end{aligned}$$

The difference of (2.15) and (4.24) of IX can, though somewhat troublesome in calculation, be estimated in a similar way.

In conclusion, we remark that the whole probability of proving non-paternity against a distinguished child alone is given by

$$(2.24)\qquad \begin{aligned}I_0-J_0=&2S_2-3S_3+S_4-8S_2S_3+8S_5\\&-4S_2^3+\tfrac{9}{2}S_3^2+15S_2S_4-\tfrac{31}{2}S_6,\end{aligned}$$

and that that against at least one child by

$$(2.25)\qquad \begin{aligned}J_0\equiv 2I_0-J_0=&1-2S_3-2S_2^2+3S_4-5S_2S_3+5S_5\\&-4S_2^3+\tfrac{9}{2}S_3^2+15S_2S_4-\tfrac{31}{2}S_6;\end{aligned}$$

cf. (6.3) and (7.1) of IX.

—To be continued—