72. Probability-theoretic Investigations on Inheritance. XI_2. Absolute Non-Paternity.

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4. Absolute non-paternity against brethren with different fathers.

Concerning brethren with different fathers, i.e., children with a mother alone in common, analogous problems arise as in the preceding section. We first consider a problem corresponding to the one discussed in § 2 of X. Let us denote by

$$D_0(ij,\ hk) \tag{4.1}$$

the probability of an event that a brethren combination (A_{ij}, A_{hk}) *with different fathers appears and then the proof of absolute non-paternity can be established against both of them.* This is the basic quantity corresponding to (2.2) of X. The explicit expression for (4.1) can immediately be derived from (2.1) by replacing merely a factor $\sigma(ij, hk)$ by the corresponding one $\sigma_0(ij, hk)$. We thus get, corresponding to (2.2) to (2.8), the following results:

$$D_0(ii,\ ii)=\tfrac{1}{2}p_i^3(1+p_i)(1-p_i)^2, \tag{4.2}$$

$$D_0(ii,hh)=\tfrac{1}{2}p_i^2p_h^2(1-p_i-p_h)^2 \qquad (h\neq i), \tag{4.3}$$

$$D_0(ii,\ ih)=\tfrac{1}{2}p_i^2p_h(1+2p_i)(1-p_i-p_h)^2 \qquad (h\neq i), \tag{4.4}$$

$$D_0(ii,hk)=p_i^2p_hp_k(1-p_i-p_h-p_k)^2 \qquad (h,k\neq i;\ h\neq k); \tag{4.5}$$

$$D_0(ij,\ ij)=\tfrac{1}{2}p_ip_j(p_i+p_j+4p_ip_j)(1-p_i-p_j)^2 \qquad (i\neq j), \tag{4.6}$$

$$D_0(ij,\ ih)=\tfrac{1}{2}p_ip_jp_h(1+4p_i)(1-p_i-p_j-p_h)^2 \qquad (i\neq j;\ h\neq i,j), \tag{4.7}$$

$$D_0(ij,hk)=2p_ip_jp_hp_k(1-p_i-p_j-p_h-p_k)^2 \qquad (i\neq j;\ h,k\neq i,j;\ h\neq k). \tag{4.8}$$

A symmetry relation corresponding to (2.9) is valid here also:

$$D_0(ij,hk)=D_0(hk,ij) \qquad (i,j,h,k=1,\ldots,m). \tag{4.9}$$

Partial sums corresponding to (2.10) and (2.11) become

$$\begin{aligned} D_0(ii)=p_i^2(1&-3S_2+\tfrac{5}{2}S_3+S_2^2-\tfrac{3}{2}S_4\\ &-(2-3S_2+S_3)p_i+2(2-S_2)p_i^2-\tfrac{11}{2}p_i^3+\tfrac{7}{2}p_i^4), \end{aligned} \tag{4.10}$$

$$\begin{aligned} D_0(ij)=2p_ip_j(1&-3S_2+\tfrac{5}{2}S_3+S_2^2-\tfrac{3}{2}S_4\\ &-(2-3S_2+S_3)(p_i+p_j)+2(2-S_2)(p_i^2+p_j^2)-2p_ip_j\\ &-\tfrac{11}{2}(p_i^3+p_j^3)-3p_ip_j(p_i+p_j)+\tfrac{7}{2}(p_i^4+p_j^4)+p_ip_j(p_i^2+p_j^2)\\ &+2p_i^2p_j^2) \qquad (i\neq j). \end{aligned} \tag{4.11}$$

Sub-probabilities over homo- and heterozygotic first children become

$$(4.12)\qquad \sum_{i=1}^{m} D_0(ii)=S_2-2S_3-3S_2^2+4S_4-\tfrac{11}{2}S_2S_3+\tfrac{11}{2}S_5 +S_2^3-S_3^2-\tfrac{7}{2}S_2S_4+\tfrac{7}{2}S_6,$$

$$(4.13)\qquad \sum_{i,j}{}' D_0(ij)=1-8S_2+\tfrac{29}{2}S_3+12S_2^2-\tfrac{45}{2}S_4-\tfrac{41}{2}S_2S_3+24S_5 -S_2^3+4S_3^2+\tfrac{15}{2}S_2S_4-11S_6,$$

the sum of which represents the whole probability

$$(4.14)\qquad D_0=1-7S_2+\tfrac{25}{2}S_3+9S_2^2-\tfrac{37}{2}S_4-15S_2S_3+\tfrac{37}{2}S_5 +3S_3^2+4S_2S_4-\tfrac{15}{2}S_6.$$

As illustrative examples we show here the whole probabilities in cases of ABO, Q, $Qq_\pm$ and MN blood types; the three former cases contain recessive genes. The results are as follows:

$$(4.15)\qquad D_{0ABO}=\tfrac{1}{2}pqr^2(1+r+2r^2+4pq),$$

$$(4.16)\qquad D_{0Q}=D_{0Qq\pm}=0,$$

$$(4.17)\qquad D_{0MN}=s^2t^2(1-st).$$

Inequalities corresponding to (2.21) and (2.22) of X can be verified in quite a similar manner, and further an inequality, corresponding to (2.23) of X, can also be shown:

$$(4.18)\qquad D_0\leqq D.$$

Problems corresponding to the ones stated at the end of §2 in X are now immediate. In fact, since the quantity C_0 corresponding to (1.6) of X coincides with C given in (1.6), the whole probability of proving non-paternity against a distinguished child alone is given by

$$(4.19)\qquad C_0-D_0=3S_2-\tfrac{17}{2}S_3-7S_2^2+\tfrac{31}{2}S_4+15S_2S_3-\tfrac{37}{2}S_5 -3S_3^2-4S_2S_4+\tfrac{15}{2}S_6,$$

and that against at least one child by

$$(4.20)\qquad \tilde{D}_0\equiv 2C_0-D_0=1-S_2-\tfrac{9}{2}S_3-5S_2^2+\tfrac{25}{2}S_4+15S_2S_3-\tfrac{37}{2}S_5 -3S_3^2-4S_2S_4+\tfrac{15}{2}S_6.$$

5. Absolute non-paternity of a father of a child against another child.

We now turn to a problem to determine a probability of an event that a father of first child can assert his non-paternity absolutely against second child; the brethren being supposed to possess different fathers. This is a problem corresponding to one discussed in §4 of X. We denote by

$$(5.1)\qquad E_0(hk, fg)$$

the probability of an event that a brethren combination (A_{hk}, A_{fg}), *possessing a mother alone in common, appears and then a father of first child can assert his non-paternity against second child.* The quantity (5.1) can be determined by modifying the procedure for determining (2.1) suitably, namely, by making use of the σ_0's instead of the σ's and the probabilities a posteriori of father of first child instead of general frequencies.

Symmetry relation similar to (2.9) will not hold in general, but an identical relation

$$(5.2)\qquad E_0(fg, fg)=0$$

does hold good; cf. (4.3) of X.

Probability a posteriori of a type A_{ab} of father possessing first child A_{hk} becomes, as already noticed in (1.28) of IV,

$$(5.3)\qquad \pi(ab; hk)/A_{hk}.$$

Thus, the results can be derived as follows:

$$(5.4)\qquad E_0(hh,\ ff)=\tfrac{1}{2}p_h^2p_f^2(1-p_f) \qquad (h\neq f),$$

$$(5.5)\qquad E_0(hf,\ ff)=\tfrac{1}{4}p_f^2p_h(1+2p_f)(1-p_f) \qquad (h\neq f),$$

$$(5.6)\qquad E_0(hk,\ ff)=p_f^2p_hp_k(1-p_f) \qquad (h,\ k\neq f;\ h\neq k);$$

$$(5.7)\qquad E_0(ff,\ fg)=0,$$

$$(5.8)\qquad E_0(hh, fg)=p_h^2p_fp_g(1-p_f-p_g) \qquad (f\neq g;\ h\neq f,\ g),$$

$$(5.9)\qquad E_0(hf, fg)=\tfrac{1}{4}p_hp_fp_g(1+4p_f)(1-p_f-p_g) \qquad (f\neq g;\ h\neq f,\ g),$$

$$(5.10)\qquad E_0(hk, fg)=2p_hp_kp_fp_g(1-p_f-p_g) \quad (f\neq g;\ h,\ k\neq f,\ g;\ h\neq k).$$

The relation (5.7) would, together with (5.2), also previously be noticed. In fact, father of homozygotic first child A_{ff} must contain at least one gene A_f and hence cannot assert his non-paternity absolutely against second child possessing this gene.

Several partial sums or sub-probabilities are obtained in the following forms:

$$(5.11)\qquad \sum_{h\neq f}(E_0(hh,\ ff)+E_0(hf,\ ff))+\sum_{h,k\neq f}{}' E_0(hk,\ ff)=\tfrac{3}{4}p_f^2(1-p_f)^2,$$

$$(5.12)\qquad \sum_{h\neq f,g}(E_0(hh, fg)+E_0(hf, fg)+E_0(hg, fg))+\sum_{h,\ k\neq f,\ g}{}' E_0(hk, fg)$$
$$=\tfrac{3}{2}p_fp_g(1-p_f-p_g)^2 \qquad (f\neq g);$$

$$(5.13)\qquad \sum_{f=1}^{m}\tfrac{3}{4}p_f^2(1-p_f)^2=\tfrac{3}{4}(S_2-2S_3+S_4),$$

$$(5.14)\qquad \sum_{f,\ g}{}'\tfrac{3}{2}p_fp_g(1-p_f-p_g)^2=\tfrac{3}{4}(1-5S_2+6S_3+2S_2^2-4S_4).$$

The sum of the last two expressions implies the *whole probability of absolute non-paternity of father of first child against second child*:

$$(5.15)\qquad E_0=\tfrac{3}{4}-3S_2+3S_3+\tfrac{3}{2}S_2^2-\tfrac{9}{4}S_4.$$

In particular case $m=2$, realized by *MN* blood type, the whole probability reduces to

$$(5.16)\qquad E_{0MN}=\tfrac{3}{2}s^2t^2.$$

The case where recessive genes are existent can be discussed similarly, what will be illustrated by an example of *ABO* blood type. Second children of O or AB alone are to be considered. In the former case, father AB of first child is deniable. First child is then either of types except O, and probabilities a posteriori of father AB, when first child is A, B, AB, are given by

$$(5.17)\quad \begin{gathered} \Pi(AB;A)/\bar{A}=\frac{q(p+r)}{p+2r},\ \Pi(AB;B)/\bar{B}=\frac{p(q+r)}{q+2r}, \\ \Pi(AB;AB)/\overline{AB}=\frac{p+q}{2}, \end{gathered}$$

respectively. Hence, corresponding to (5.11), we get

$$(5.18)\quad \begin{aligned} &\tfrac{1}{2}pr^2(1+p+2r)\frac{q(p+r)}{p+2r}+\tfrac{1}{2}qr^2(1+q+2r)\frac{p(q+r)}{q+2r}+pqr^2\frac{p+q}{2} \\ &=pqr^2\Big(1+\frac{1}{2}\frac{p+r}{p+2r}+\frac{1}{2}\frac{q+r}{q+2r}\Big). \end{aligned}$$

In the latter case, father O of first child is deniable. First child is then either of types except AB, and probabilities a posteriori of father O, when first child is O, A, B, are given by

$$(5.19)\quad \Pi(O;O)/\bar{O}=r,\ \Pi(O;A)/\bar{A}=\frac{r^2}{p+2r},\ \Pi(O;B)/\bar{B}=\frac{r^2}{q+2r},$$

respectively. Hence, corresponding to (5.12), we get

$$(5.20)\quad \begin{aligned} &pqr^2r+\frac{1}{2}pq(p+r+2p^2+4pr)\frac{r^2}{p+2r}+\frac{1}{2}pq(q+r+2q^2+4qr)\frac{r^2}{q+2r} \\ &=pqr^2\Big(1+\frac{1}{2}\frac{p+r}{p+2r}+\frac{1}{2}\frac{q+r}{q+2r}\Big). \end{aligned}$$

The whole probability is, as a sum of (5.18) and (5.20), expressed in the form

$$(5.21)\quad E_{0ABO}=pqr^2\Big(2+\frac{p+r}{p+2r}+\frac{q+r}{q+2r}\Big).$$

Evidently, the corresponding probabilities on Q and $Qq_{\pm}$ blood types vanish:

$$(5.22)\quad E_{0Q}=E_{0Qq\pm}=0.$$

By the way, we notice that between ABO and MN blood types the discontinuity of the same nature as in § 6 of VII is observed at several places.

6. Maximizing distributions.

Problem to determine maximizing distribution for various probabilities derived in the present chapter can be discussed by usual manner.

The probability C_{ABO} given in (1.7) is maximized by the distribution

$$(6.1)\quad p=q=1/4,\ r=1/2;\ \bar{O}=1/4,\ \bar{A}=\bar{B}=5/16,\ \overline{AB}=1/8;$$

the maximum being

(6.2) $$(C_{ABO})^{\max}=1/16=0.0625.$$

The probability C_{MN} given in (1.9) is maximized by the distribution

(6.3) $$s=t=1/2;\quad \bar{M}=\bar{N}=1/4,\ \overline{MN}=1/2;$$

the maximum being

(6.4) $$(C_{MN})^{\max}=1/8=0.1250.$$

The probability (1.6) for general case attains, for the symmetric distribution

(6.5) $$p_i=1/m \qquad (i=1, \dots, m),$$

its stationary value given by

(6.6) $$(C)^{\mathrm{stat}}=(1-1/m)(1-3/m+3/m^2),$$

which would perhaps be the actual maximum as is the case for $m=2$.

Next, the probability D_{ABO} given in (2.15) is shown to attain its maximum for the distribution

(6.7) $$p=q=0.2375,\quad r=0.5250;$$
$$\bar{O}=0.2756,\quad \bar{A}=\bar{B}=0.3058,\quad \overline{AB}=0.1128,$$

where the coinciding value of p and q is a root of a cubic equation

(6.8) $$36x^3-42x^2+29x-5=0;$$

the maximum being

(6.9) $$(D_{ABO})^{\max}=0.0263.$$

The probability D_{MN} given in (2.17) is maximized again by the distribution (6.3); the maximum being

(6.10) $$(D_{MN})^{\max}=9/128=0.0703.$$

The probability (2.14) for general case attains, again for the distribution (6.5), its stationary value

(6.11) $$(D)^{\mathrm{stat}}=(1-1/m)(1-5/m+12/m^2-37/4m^3+15/4m^4),$$

which would perhaps be the acutal maximum.

Next, the probability D_{0ABO} given in (4.15) is maximized by the distribution

(6.12) $$p=q=0.2194,\quad r=0.5612;$$
$$\bar{O}=0.3149,\quad \bar{A}=\bar{B}=0.2944,\quad \overline{AB}=0.0963,$$

where the coinciding value of p and q is a root of a cubic equation

(6.13) $$72x^3-74x^2+31x-4=0;$$

the maximum of (4.15) being

(6.14) $$(D_{0ABO})^{\max}=0.0180.$$

The probability D_{0MN} given in (4.17) is maximized again by the distribution (6.3); the maximum being

(6,15) $$(D_{0MN})^{\max}=3/64=0.0469.$$

The probability (4.14) for general case attains, again for the distribution (6.5), its stationary value

(6.16) $$(D_0)^{\mathrm{stat}}=(1-1/m)^2(1-5/m+21/2m^2-15/2m^3),$$

which would be expected to be the actual maximum.

The corresponding problems on the probabilities $C-D$ and $\tilde{D} \equiv 2C-D$ as well as C_0-D_0 and $\tilde{D}_0 \equiv 2C_0-D_0$ will be left to the reader.

Last, we consider the probabilities derived in the preceding section. The probability given in (5.16) attains its maximum again for the distribution (6.3); the maximum being

(6.17) $$(E_{0MN})^{\max}=3/32=0.09375.$$

The value of (5.15) for the distribution (6.5) becomes

(6.18) $$(E_0)^{\text{stat}}=(3/4)(1-1/m)(1-3/m+3/m^2).$$

It would be noticed that the comparison between (1.6) and (5.15) (or, rather precisely, between (1.2), (1.3) and (5.11), (5.12)) implies *a remarkable relation*

(6.19) $$E_0=\frac{3}{4}C.$$

Consequently, as $m\to\infty$, E_0 tends to 3/4 while C to 1. However, in case of ABO blood type where a recessive gene is existent, such an identity does not hold. In fact, comparing (1.7) with (5.21), we get

$$E_{0ABO}-\frac{3}{4}C_{ABO}=\frac{1}{2}pqr^2\left(\frac{p}{p+2r}+\frac{q}{q+2r}\right),$$

$$C_{ABO}-E_{0ABO}=pqr^2\left(\frac{1}{p+2r}+\frac{1}{q+2r}\right),$$

and hence the inequalities

(6.20) $$C_{ABO}>E_{0ABO}>\frac{3}{4}C_{ABO},$$

equality signs being excluded since the trivial distributions may be rejected.

We observe in (6.19) or (6.20) that, given a child, the non-paternity proof is expected probabilistically at less rate by a father of a brother of the given child than by a man chosen at random. The deficiency of E_0, compared with C, may be regarded as being caused by a positive correlation, intermediated by a common mother, between a type of the given child and possible types of a father of another child.

In conclusion, we remark that the *problems of proving absolute non-maternity* are the quite same as those on non-paternity at least from the probabilistic view-point. Non-maternity problems would be expected, for instance, when, in a case of succession to a property after death of father, a woman must be judged whether she is a true mother of a left child or not.