83. Probability-theoretic Investigations on Inheritance. XII$_1$. Probability of Paternity.

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1. Problem and its background.

In several preceding chapters[1] we discussed various problems on non-paternity. A typical problem concerned the case where a man in question could prove his non-paternity against a given child, based upon inherited characters under consideration. Distinction was made according to that a character of child alone, or also of its mother and its brother was known. In every problem, non-paternity proof was based upon the fact that the characters were incompatible, and hence the proof was predicative provided that the inheritance mode was so established.

On the other hand, if the cases where non-paternity proof is successful are excluded, the remaining cases cannot be predicatively decided, at least based upon the inherited character alone. In other words, if a type of a man is compatible with that of a given child, then non-paternity proof is of course impossible, but this impossibility does not necessarily mean that the paternity is affirmatively proved. However, there will arise a problem to estimate whether in such cases the paternity is or is not to a certain degree probable based on the inherited characters. In the present chapter we shall investigate such a problem. The main tool of attack is the *Bayes' theorem* referred to at the end of § 1 in IV[2].

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1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethren combinations; VI. Rate of danger in random blood transfusion; VII. Non-paternity problems; VIII. Further discussions on non-paternity; IX. Non-paternity problems concerning mother-children combinations; X. Non-paternity problems concerning mother-child-child combinations. XI. Absolute non-paternity. Proc. Japan Acad., **27** (1951), I. 371-377; II. 378-383, 384-387; III. 459-465, 466-471, 472-477, 478-483; IV. 587-592, 593-597, 598-603, 605-610, 611-614, 615-620; V. 689-693, 694-699; **28** (1952), VI. 54-58. VII. 102-104, 105-108, 109-111, 112-115, 116-120, 121-125; VIII. 162-164, 165-168, 169-171; IX. 207-212, 213-217, 218-223, 224-229; X. 249-253, 254-258, 259-264; XI. 311-316, 317-322. These papers will be referred to as I; II; III; IV; V; VI; VII; VIII; IX; X; XI.

2) Such a problem has been compendiously discussed in E. Essen-Möller und C.-E. Quensel, Zur Theorie des Vaterschaftsbeweises auf Grund von Ähnlichkeitsbefunden, Deut. Zeitschr. f. ges. gerichtl. Med. **31** (1939), 70-96. Cf. also Y. Komatu, On decision of paternity based upon blood types (Japanese), Hanzaigaku-Zasshi **13** (1939), 485-494.

Now, given a mother-child combination, we shall determine a probability a posteriori of a man whose paternity proof is the problem at issue. We call it briefly a *probability of paternity*. Two mutually exclusive causes are considered; namely, the one is that a presented man is really a father of the child and the other is that he is not its father, which will be denoted by C_1 and C_2, respectively. If there is no information as to paternity before inherited characters are known, it will be reasonable to suppose that *the probabilities a priori of both causes are equal each other, i.e., they are equal to* 1/2.

Let a fixed mother-child combination $(A_{ij}; A_{hk})$ be given. The probability of an event that, under the cause C_1, a mating $A_{ab}\times A_{ij}$ produces a child A_{hk} has already been calculated essentially. In fact, a table of § 3 in I remains of service here also; the value listed there has only to be divided by the corresponding mating probability. It will be denoted by

$$\lambda(ab, ij; hk). \tag{1.1}$$

On the other hand, under the cause C_2, a mother A_{ij} produces a child A_{hk} with the probability

$$\pi(ij; hk)/\bar{A}_{ij}. \tag{1.2}$$

Here, $\pi(ij; hk)$ denotes the probability of mother-child combination defined in (1.1) of IV and the quantity (1.2) itself has been introduced in (1.27) of IV. Hence, the Bayes' theorem implies that, given a mother-child combination $(A_{ij}; A_{hk})$, the probability a posteriori of a man A_{ab} to be a true father, i.e., his probability of paternity, is expressed by

$$\Lambda(ij; hk; ab)=\frac{\lambda(ab, ij; hk)}{\lambda(ab, ij; hk)+\pi(ij; hk)/\bar{A}_{ij}}. \tag{1.3}$$

It would be remarked that the notion of probability of paternity belongs, in nature, to quite a different category from that of non-paternity.

In particular, the non-paternity proof can be established if and only if the mating $A_{ab}\times A_{ij}$ does never produce a child A_{hk}. The probability (1.1), i.e., the numerator of the right-hand side of (1.3), is then regarded as to be equal to zero, and hence the probability of paternity (1.3) vanishes. We have supposed that the probability a priori of either cause C_1 or C_2 is equal to 1/2. Hence, the paternity of a man in question is probable or not according to that the thus defined probability (1.3) is near to 1 or to 0.

2. Probability of paternity.

We shall now calculate the value of (1.3) for every possible triple.

A mother of homozygote A_{ii} can produce a child of any type containing at least a gene A_i, i.e., A_{ii} or A_{ih} $(h \neq i)$, among which a child A_{ii} must possess a father containing at least a gene A_i, i.e., of a type A_{ii} or A_{ik} $(k \neq i)$. The formula (1.3) yields thus

$$(2.1) \qquad \Lambda(ii; ii; ii) = 1/(1+p_i),$$

$$(2.2) \qquad \Lambda(ii; ii; ik) = \tfrac{1}{2}/(\tfrac{1}{2}+p_i) = 1/(1+2p_i) \qquad (k \neq i).$$

Comparison between (2.1) and (2.2) shows that, for a mother-child combination $(A_{ii}; A_{ii})$, paternity of a man A_{ii} seems more probable than of a man A_{ik} $(k \neq i)$. A child A_{ih} $(h \neq i)$ of a mother A_{ii} must possess a father containing at least a gene A_h, i.e., either of a type A_{ih}, A_{hh}, A_{hk} $(k \neq i, h)$. From formula (1.3), we get

$$(2.3) \qquad \Lambda(ii; ih; ih) = \tfrac{1}{2}/(\tfrac{1}{2}+p_h) = 1/(1+2p_h) \qquad (h \neq i),$$

$$(2.4) \qquad \Lambda(ii; ih; hh) = 1/(1+p_h) \qquad (h \neq i),$$

$$(2.5) \qquad \Lambda(ii; ih; hk) = \tfrac{1}{2}/(\tfrac{1}{2}+p_h) = 1/(1+2p_h) \qquad (h \neq i,\ k \neq i, h),$$

whence we see that, for mother-child combination $(A_{ii}; A_{ih})$ $(h \neq i)$, the most probable type of father is A_{hh}. A mother of a fixed heterozygote A_{ij} $(i \neq j)$ can produce a child of any type containing at least one of genes A_i and A_j, i.e., A_{ii}, A_{jj}, A_{ij}, A_{ih}, A_{jh} $(h \neq i, j)$. Among those a child A_{ii} must possess a father containing at least a gene A_i, and hence we get

$$(2.6) \qquad \Lambda(ij; ii; ii) = \tfrac{1}{2}/(\tfrac{1}{2}+\tfrac{1}{2}p_i) = 1/(1+p_i),$$

$$(2.7) \qquad \Lambda(ij; ii; ij) = \tfrac{1}{4}/(\tfrac{1}{4}+\tfrac{1}{2}p_i) = 1/(1+2p_i),$$

$$(2.8) \qquad \Lambda(ij; ii; ih) = \tfrac{1}{4}/(\tfrac{1}{4}+\tfrac{1}{2}p_i) = 1/(1+2p_i) \qquad (h \neq i, j),$$

and similarly the quantities obtained by interchanging the suffices i and j, whence follows that, for a homozygotic child with a heterozygotic mother, the most probable type of father is the same as that of child. For mother-child combination $(A_{ij}; A_{ij})$, the possible types of a father are those containing at least one of genes A_i and A_j. We hence get

$$(2.9) \qquad \Lambda(ij; ij; ii) = \tfrac{1}{2}/(\tfrac{1}{2}+\tfrac{1}{2}(p_i+p_j)) = 1/(1+p_i+p_j),$$

$$(2.10) \qquad \Lambda(ij; ij; ij) = \tfrac{1}{2}/(\tfrac{1}{2}+\tfrac{1}{2}(p_i+p_j)) = 1/(1+p_i+p_j),$$

$$(2.11) \qquad \Lambda(ij; ij; ih) = \tfrac{1}{4}/(\tfrac{1}{4}+\tfrac{1}{2}(p_i+p_j)) = 1/(1+2(p_i+p_j)) \qquad (h \neq i, j);$$

and similarly the quantities obtained by interchanging the suffices i and j, whence follows that, for a mother-child combination $(A_{ij}; A_{ij})$, the types A_{ii}, A_{jj} and A_{ij} are more probable as father's type than any other types. A child A_{ih} $(h \neq i, j)$ with a mother A_{ij} must possess a father containing at least a gene A_h. We thus get

$$(2.12)\qquad A(ij;\,ih;\,ih)=\tfrac{1}{4}/(\tfrac{1}{4}+\tfrac{1}{2}p_h)=1/(1+2p_h)\qquad (h\neq i,j),$$

$$(2.13)\qquad A(ij;\,ih;\,jh)=\tfrac{1}{4}/(\tfrac{1}{4}+\tfrac{1}{2}p_h)=1/(1+2p_h)\qquad (h\neq i,j),$$

$$(2.14)\qquad A(ij;\,ih;\,hh)=\tfrac{1}{2}/(\tfrac{1}{2}+\tfrac{1}{2}p_h)=1/(1+p_h)\qquad (h\neq i,j),$$

$$(2.15)\qquad A(ij;\,ih;\,hk)=\tfrac{1}{4}/(\tfrac{1}{4}+\tfrac{1}{2}p_h)=1/(1+2p_h)\qquad (h\neq i,j;\ k\neq i,j,h);$$

and similarly the quantities obtained by interchanging the suffices i and j, whence follows that, for a mother-child combination consisting of different heterozygotes, the homozygote with the gene in child not common to mother is the most probable type as father's one. It is also to be noticed that the quantities (2.12) to (2.15) are all independent of suffices i and j.

All the possible combinations have thus been essentially worked out. The results are tabulated as follows, different letters being supposed, as usual, to denote different genes.

Mother	Child \ Man	A_{ii}	A_{ih}	A_{ik}	A_{hh}	A_{hk}	A_{kl}
A_{ii}	A_{ii}	$\frac{1}{1+p_i}$	$\frac{1}{1+2p_i}$	$\frac{1}{1+2p_i}$	0	0	0
	A_{ih}	0	$\frac{1}{1+2p_h}$	0	$\frac{1}{1+p_h}$	$\frac{1}{1+2p_h}$	0

Mother	Child \ Man	A_{ii}	A_{jj}	A_{ij}	A_{ih}	A_{jh}	A_{hh}	A_{hk}	A_{kl}
A_{ij}	A_{ii}	$\frac{1}{1+p_i}$	0	$\frac{1}{1+2p_i}$	$\frac{1}{1+2p_i}$	0	0	0	0
	A_{jj}	0	$\frac{1}{1+p_j}$	$\frac{1}{1+2p_j}$	0	$\frac{1}{1+2p_j}$	0	0	0
	A_{ij}	$\frac{1}{1+p_i+p_j}$	$\frac{1}{1+p_i+p_j}$	$\frac{1}{1+p_i+p_j}$	$\frac{1}{1+2(p_i+p_j)}$	$\frac{1}{1+2(p_i+p_j)}$	0	0	0
	A_{ih} or A_{jh}	0	0	0	$\frac{1}{1+2p_h}$	$\frac{1}{1+2p_h}$	$\frac{1}{1+p_h}$	$\frac{1}{1+2p_h}$	0

The above discussions have concerned genotypes. Similar results will also be derived with respect to phenotypes in case where recessive genes may also be existent. As illustrative examples, we tabulate here those on various blood types[3].

3) Cf. also Y. Komatu, loc. cit.[2]

$$x^* \equiv p^2+3pr+r^2, \qquad y^* \equiv q^2+3qr+r^2$$

Mother	Child \ Man	O	A	B	AB
O	O	$\frac{1}{1+r}$	$\frac{1}{1+p+2r}$	$\frac{1}{1+q+2r}$	0
	A	0	$\frac{p+r}{p+r+p(p+2r)}$	0	$\frac{1}{1+2p}$
	B	0	0	$\frac{q+r}{q+r+q(q+2r)}$	$\frac{1}{1+2q}$
A	O	$\frac{1}{1+r}$	$\frac{1}{1+p+2r}$	$\frac{1}{1+q+2r}$	0
	A	$\frac{p+r}{p+r+x^*}$	$\frac{(p+r)(p+3r)}{(p+r)(p+3r)+(p+2r)x^*}$	$\frac{r(p+r)}{r(p+r)+(q+2r)x^*}$	$\frac{p+2r}{p+2r+2x^*}$
	B	0	0	$\frac{q+r}{q+r+q(q+2r)}$	$\frac{1}{1+2q}$
	AB	0	0	$\frac{q+r}{q+r+q(q+2r)}$	$\frac{1}{1+2q}$
B	O	$\frac{1}{1+r}$	$\frac{1}{1+p+2r}$	$\frac{1}{1+q+2r}$	0
	A	0	$\frac{p+r}{p+r+p(p+2r)}$	0	$\frac{1}{1+2p}$
	B	$\frac{q+r}{q+r+y^*}$	$\frac{r(q+r)}{r(q+r)+(p+2r)y^*}$	$\frac{(q+r)(q+3r)}{(q+r)(q+3r)+(q+2r)y^*}$	$\frac{q+2r}{q+2r+2y^*}$
	AB	0	$\frac{p+r}{p+r+p(p+2r)}$	0	$\frac{1}{1+2p}$
AB	A	$\frac{1}{1+p+r}$	$\frac{1}{1+p+r}$	$\frac{r}{r+(p+r)(q+2r)}$	$\frac{1}{1+2(p+r)}$
	B	$\frac{1}{1+q+r}$	$\frac{r}{r+(q+r)(p+2r)}$	$\frac{1}{1+q+r}$	$\frac{1}{1+2(q+r)}$
	AB	0	$\frac{p+r}{p+r+(p+q)(p+2r)}$	$\frac{q+r}{q+r+(p+q)(q+2r)}$	$\frac{1}{1+p+q}$

Mother	Child \ Man	Q	q
Q	Q	$\frac{1+2v}{1+2v+(1+v)(1+uv)}$	$\frac{1}{2+uv}$
	q	$\frac{1}{2+v}$	$\frac{1}{1+v}$
q	Q	$\frac{1}{1+u(1+v)}$	0
	q	$\frac{1}{2+v}$	$\frac{1}{1+v}$

Mother	Child \ Man	Q	q_-	q_+
Q	Q	$\frac{1+2v}{1+2v+(1+v)(1+uv)}$	$\frac{1}{2+uv}$	$\frac{1}{2+uv}$
	q_-	$\frac{1}{2+v}$	$\frac{v^2+v_1v_2}{v^2+v_1v_2+v_1(v+v_2)^2}$	$\frac{1}{1+v+v_2}$
	q_+	$\frac{1}{2+v}$	$\frac{1}{1+v+v_2}$	$\frac{1}{1+v_2}$
q_-	Q	$\frac{1}{1+u(1+v)}$	0	0
	q_-	$\frac{1}{2+v}$	$\frac{v(v+2v_2)}{v(v+2v_2)+(v+v_2)(v^2+v_1v_2)}$	$\frac{v}{v+v^2+v_1v_2}$
	q_+	$\frac{1}{2+v}$	$\frac{1}{1+v+v_2}$	$\frac{1}{1+v_2}$
q_+	Q	$\frac{1}{1+u(1+v)}$	0	0
	q_-	$\frac{1}{2+v}$	$\frac{v}{v+v_1(v+v_2)}$	0
	q_+	$\frac{1}{2+v}$	$\frac{1}{1+v+v_2}$	$\frac{1}{1+v_2}$

—*To be continued*—