100. *Probability-theoretic Investigations on Inheritance.* *$XIII_2$. Estimation of Genotypes.*

By Yûsaku KOMATU.

Department of Mathematics, Tokyo Institute of Technology and
Department of Legal Medicine, Tokyo Medical and Dental University.

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3. Estimation without reference to spouse.

The problem discussed in § 2 concerned the case where the type of a spouse of an individual in question is also taken into account. The corresponding problem may be treated independently of the type of a spouse.

We first consider again the simplest case, *the Q blood type*. Let an individual of phenotype Q be given. Then, the type q of its child is impossible unless the individual is heterozygotic. Hence, we have only to consider the case where all the n children of the individual are of the type Q. In this case, we denote by

$$\Pr\{Q=QQ|\to Q^n\} \quad \text{and} \quad \Pr\{Q=Qq|\to Q^n\}$$

the probabilities a posteriori of the individual to be of homozygote QQ and of heterozygote Qq, respectively, which will be determined in the following lines.

Now, the probabilities a priori of QQ and Qq among Q are regarded as $\overline{QQ}/\overline{Q}=u/(1+v)$ and $\overline{Qq}/\overline{Q}=2v/(1+v)$, respectively, the ratio being $u:2v$. An individual QQ produces Q alone, while an individual Qq produces Q with probability

$$\frac{\pi(Qq;QQ)+\pi(Qq;Qq)}{\overline{Qq}}=\frac{1+u}{2},$$

the π's denoting the probabilities of mother-child combinations defined in § 1 of IV, which may also be regarded as those of father-child combinations. Thus, based on the Bayes' theorem, we get the desired probabilities

$$(3.1) \qquad \Pr\{Q=QQ|\to Q^n\}=\frac{u\cdot 1^n}{u\cdot 1^n+2v\left(\frac{1+u}{2}\right)^n}=\frac{2^{n-1}u}{2^{n-1}u+v(1+u)^n},$$

$$(3.1') \qquad \Pr\{Q=Qq|\to Q^n\}=1-\Pr\{Q=QQ|\to Q^n\}=\frac{v(1+u)^n}{2^{n-1}u+v(1+u)^n}.$$

We proceed to deal with *the ABO blood type*. Let an individual of phenotype A be given. If it is homozygotic, then the type of a child is restricted to A or AB, while if it is heterozygotic, then any type of a child is possible. Accordingly, if there exists at least one child of O or B, then the individual must be heterozygotic. Hence, we have only to consider the case, where all the children are of the type A or AB. If, among all the n children, there are ν children A and $n-\nu$ children AB, we denote by $\Pr\{A=AA|\to A^{\nu}\frown AB^{n-\nu}\}$ and $\Pr\{A=AO|\to A^{\nu}\frown AB^{n-\nu}\}$ the probabilities a posteriori of the individual to be homozygotic and heterozygotic, respectively. Now, the probabilities a priori of AA and AO have a ratio $p:2r$. An individual AA produces A and AB with respective probabilities

$$\frac{\pi(AA;AA)+\pi(AA;AO)}{\overline{AA}}=p+r \quad \text{and} \quad \frac{\pi(AA;AB)}{\overline{AA}}=q,$$

while an individual AO produces A and AB with respective probabilities

$$\frac{\pi(AO;AA)+\pi(AO;AO)}{\overline{AO}}=\frac{2p+r}{2} \quad \text{and} \quad \frac{\pi(AO;AB)}{\overline{AO}}=\frac{q}{2}.$$

Thus, we obtain, by means of the Bayes' theorem, the desired probabilities

$$\text{(3.2)} \quad \Pr\{A=AA|\to A^{\nu}\frown AB^{n-\nu}\}$$
$$=\frac{p(p+r)^{\nu}q^{n-\nu}}{p(p+r)^{\nu}q^{n-\nu}+2r\left(\frac{2p+r}{2}\right)^{\nu}\left(\frac{q}{2}\right)^{n-\nu}}=\frac{2^{n-1}p(p+r)^{\nu}}{2^{n-1}p(p+r)^{\nu}+r(2p+r)^{\nu}},$$

$$\text{(3.2}'\text{)} \quad \Pr\{A=AO|\to A^{\nu}\frown AB^{n-\nu}\}$$
$$=1-\Pr\{A=AA|\to A^{\nu}\frown AB^{n-\nu}\}=\frac{r(2p+r)^{\nu}}{2^{n-1}p(p+r)^{\nu}+r(2p+r)^{\nu}}.$$

The corresponding probabilities with respect to an individual of type B can be immediately written down. In fact, we have only to replace A, B, p by B, A, q, respectively. We thus get, corresponding to (3.2) and (3.2′), the following expressions

$$\text{(3.3)} \quad \Pr\{B=BB|\to B^{\nu}\frown AB^{n-\nu}\}=\frac{2^{n-1}q(q+r)^{\nu}}{2^{n-1}q(q+r)^{\nu}+r(2q+r)^{\nu}},$$

$$\text{(3.3}'\text{)} \quad \Pr\{B=BO|\to B^{\nu}\frown AB^{n-\nu}\}=\frac{r(2q+r)^{\nu}}{2^{n-1}q(q+r)^{\nu}+r(2q+r)^{\nu}}.$$

By the way, if all the n children are known merely as either A or AB, then the probabilities a posteriori of the individual to be homozygotic and heterozygotic are given respectively by

$$\text{(3.4)} \quad \Pr\{A=AA|\to(A\cup AB)^{n}\}$$
$$=\frac{p\cdot 1^{n}}{p\cdot 1^{n}+2r\left(\frac{1+p}{2}\right)^{n}}=\frac{2^{n-1}p}{2^{n-1}p+r(1+p)^{n}},$$

$$\text{(3.4}'\text{)} \quad \Pr\{A=AO|\to(A\cup AB)^{n}\}$$
$$=1-\Pr\{A=AA|\to(A\cup AB)^{n}\}=\frac{r(1+p)^{n}}{2^{n-1}p+r(1+p)^{n}},$$

since an individual AA produces A or AB with probability

$$\frac{\pi(AA;AA)+\pi(AA;AO)+\pi(AA;AB)}{\overline{AA}}=p+r+q=1,$$

while an individual AO produces A or AB with probability

$$\frac{\pi(AO;AA)+\pi(AO;AO)+\pi(AO;AB)}{\overline{AA}}=\frac{p}{2}+\frac{p+r}{2}+\frac{q}{2}=\frac{1+p}{2}.$$

Similarly, we get, by interchanging A and B, the corresponding probabilities

$$(3.5)\qquad \Pr\{B=BB|\rightarrow(B\cup AB)^n\}=\frac{2^{n-1}q}{2^{n-1}q+r(1+q)^n},$$

$$(3.5')\qquad \Pr\{B=BO|\rightarrow(B\cup AB)^n\}=\frac{r(1+q)^n}{2^{n-1}q+r(1+q)^n}.$$

Comparing (3.4) and (3.5) with (3.2) and (3.3) respectively, we notice that the inequalities

$$(3.6)\qquad \begin{cases} \Pr\{A=AA|\rightarrow A^n\}<\Pr\{A=AA|\rightarrow(A\cup AB)^n\} \\ \qquad\qquad\qquad\qquad <\Pr\{A=AA|\rightarrow AB^n\}, \\ \Pr\{B=BB|\rightarrow B^n\}<\Pr\{B=BB|\rightarrow(B\cup AB)^n\} \\ \qquad\qquad\qquad\qquad <\Pr\{B=BB|\rightarrow AB^n\} \end{cases}$$

hold good except for the trivial distributions with $pqr=0$, which correspond to (2.15).

The corresponding estimations can be made for other inherited characters in quite a similar way. We give here, making use of the notations similarly understood as above, the results on *the* $Qq_\pm$ *blood* type.

$$(3.7)=(3.1)\qquad \Pr\{Q=QQ|\rightarrow Q^n\}=\frac{2^{n-1}u}{2^{n-1}u+v(1+u)^n},$$

$$(3.7')\qquad \Pr\{Q=Qq_-|\rightarrow Q^n\}=\frac{v_1(1+u)^n}{2^{n-1}u+v(1+u)^n},$$

$$(3.7'')\qquad \Pr\{Q=Qq_+|\rightarrow Q^n\}=\frac{v_2(1+u)^n}{2^{n-1}u+v(1+u)^n};$$

$$(3.8')\qquad \Pr\{Q=Qq_-|\rightarrow Q^\nu\cap q_-^{n-\nu}\}=\frac{v^{n-\nu}}{v^{n-\nu}+v_2v_1^{n-\nu-1}}\qquad(\nu<n),$$

$$(3.8'')\qquad \Pr\{Q=Qq_+|\rightarrow Q^\nu\cap q_-^{n-\nu}\}=\frac{v_2v_1^{n-\nu-1}}{v^{n-\nu}+v_2v_1^{n-\nu-1}}\qquad(\nu<n);$$

$$(3.9)\qquad \Pr\{q_-=q_-q_-|\rightarrow q_-^\nu\cap Q^{n-\nu}\}=\frac{2^{\nu-1}v_1v^\nu}{2^{\nu-1}v_1v^\nu+v_2(v+v_1)^\nu},$$

$$(3.9')\qquad \Pr\{q_-=q_-q_+|\rightarrow q_-^\nu\cap Q^{n-\nu}\}=\frac{v_2(v+v_1)^\nu}{2^{\nu-1}v_1v^\nu+v_2(v+v_1)^\nu}.$$

By the way, we notice finally the further probabilities

$$(3.10)\quad \Pr\{Q=QQ|\to(Q\cup q_-)^n\}=\frac{2^{n-1}u}{2^{n-1}(u+2v_1)+v_2(1+u+v_1)^n},$$

$$(3.10')\quad \Pr\{Q=Qq_-|\to(Q\cup q_-)^n\}=\frac{2^n v_1}{2^{n-1}(u+2v_1)+v_2(1+u+v_1)^n},$$

$$(3.10'')\quad \Pr\{Q=Qq_+|\to(Q\cup q_-)^n\}=\frac{v_2(1+u+v_1)^n}{2^{n-1}(u+2v_1)+v_2(1+u+v_1)^n};$$

$$(3.11)\quad \Pr\{q_-=q_-q_-|\to(q_-\cup Q)^n\}=\frac{2^{n-1}v_1}{2^{n-1}v_1+v_2(1+u+v_1)^n},$$

$$(3.11')\quad \Pr\{q_-=q_-q_+|\to(q_-\cup Q)^n\}=\frac{v_2(1+u+v_1)^n}{2^{n-1}v_1+v_2(1+u+v_1)^n}.$$

4. Lower bound for number of children.

The probability a posteriori of an individual representing a dominant character to be homozygotic has been computed in the preceding sections with or without reference to type of its spouse. Applying the results obtained, we shall now deal with a problem stated as follows: *Given an individual representing a dominant character, how many children of the same type as that of the individual will suffice to presume the type of the individual to be homozygotic with a probability not less than a preassigned value?* A lower bound for the number of children will be obtained by solving an inequality stating that the respective probability a posteriori is not less than the preassigned value.

Let the preassigned value be denoted by α with $0<\alpha<1$. For the case $\{Q=QQ|\times Q\to Q^n\}$ in (2.1), the inequality

$$\Pr\{Q=QQ|\times Q\to Q^n\}\geqq\alpha$$

is solved by

$$(4.1)\qquad n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2v}{u}\right)\Big/\log\frac{2(1+v)}{2+v}.$$

For instance, if $u=1/5$ and $v=4/5$, it becomes

$$n\geqq\log\frac{8\alpha}{1-\alpha}\Big/\log\frac{9}{7},$$

and further if we put $\alpha=9/10$ or $\alpha=4/5$, we get respectively

$$n\geqq\log 72/(\log 9-\log 7)=17.02\ldots,$$

$$n\geqq\log 32/(\log 9-\log 7)=13.79\ldots.$$

Thus, in case, $u=1/5$ and $v=4/5$, if an individual Q accompanied by its spouse Q has produced the children of Q alone, then it may be presumed to be homozygotic with a probability greater than 9/10 or 4/5 provided the number of children exceeds 17 or 13 respectively. These bounds will be perhaps too large for a practical use, but a smaller bound will be found by conceding the probability of confidence.

For the case in (2.2), the inequality $\Pr\{Q=QQ|\times q\to Q^n\}\geqq\alpha$ is solved by

$$n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2v}{u}\right)\Big/\log 2. \tag{4.2}$$

For instance, if $u=1/5$ and $v=4/5$, it becomes

$$n\geqq\log\frac{8\alpha}{1-\alpha}\Big/\log 2,$$

and further if we put $\alpha=9/10$ or $\alpha=4/5$, we get respectively

$$n\geqq\log 72/\log 2=6.17\ldots,\qquad n\geqq\log 32/\log 2=5.$$

Thus, in case $u=1/5$ and $v=4/5$, if an individual Q accompanied by its spouse q has produced the children Q alone, then it may be presumed to be homozygotic with a probability greater than 9/10 or not less than 4/5 provided the number of children exceeds 6 or is not less than 5, respectively.

In a similar way, we obtain, for the cases (2.3) to (2.5) concerning the ABO blood type, the solution of the corresponding inequalities respectively as follows:

$$n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log 2, \tag{4.3}$$

$$n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log\frac{2(p+2r)}{2p+3r}, \tag{4.4}$$

$$n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log 2, \tag{4.5}$$

while we get, by solving a corresponding inequality for the case (2.6), an inequality

$$n-\nu\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log 2. \tag{4.6}$$

However, in case $\nu=n$, it would here be noticed that, since the probability for $\nu=n$, i.e., $\Pr\{A=AA|\times AB\to A^n\}=p/(p+2r)$ is independent of the value of n, the inequality for $\nu=n$ does always or does never hold provided the right-hand member of (4.6) is non-positive or positive, respectively.

Similar results can also be derived for the case (2.7) to (2.10). In fact, we have only to replace p by q in (4.3) to (4.6), respectively.

The solutions of the corresponding inequalities for the cases (2.11), (2.12) become respectively

$$n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log 2,\quad n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log\frac{4}{3}. \tag{4.7}$$

The results on the $Qq_{\pm}$ blood type will also be derived from (2.16) to (2.26); the actual calculation will be left to the reader.

We shall now proceed to consider the corresponding problem on the probabilities a posteriori given in § 3. First, for the case $\{Q=QQ|\to Q^n\}$ in (3.1), the inequality $\Pr\{Q=QQ|\to Q^n\}\geqq\alpha$ is solved by

$$n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2v}{u}\right)\Big/\log\frac{2}{1+u}. \tag{4.8}$$

For the case in (3.2), the inequality $\Pr\{A=AA|\to A^{\nu}\cap AB^{n-\nu}\}\geqq\alpha$ is solved in the form

$$n\geqq\nu\log\frac{2p+r}{p+r}\Big/\log 2+\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log 2. \tag{4.9}$$

In particular cases $\nu=n$ and $\nu=0$, the last inequality becomes respectively

$$\begin{cases} n\geqq\log\left(\dfrac{\alpha}{1-\alpha}\dfrac{2r}{p}\right)\Big/\log\dfrac{2(p+r)}{2p+r}, \\ n\geqq\log\left(\dfrac{\alpha}{1-p}\dfrac{2r}{p}\right)\Big/\log 2. \end{cases} \tag{4.9$'$}$$

Similar results will also be derived with respect to (3.3).

We obtain the solution of the corresponding inequality for the case (3.4) in the form

$$n\geqq\log\left(\frac{\alpha}{1-\alpha}\frac{2r}{p}\right)\Big/\log\frac{2}{1+p}, \tag{4.10}$$

and similarly for the case (3.5).

The results on the $Qq_{\pm}$ blood type are omitted, being left to the reader.