89. *A Generalization of a Theorem of Suetuna on Dirichlet Series.*

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Introduction.

Professor Z. Suetuna proved in Tôhoku Math. Journal 27, 1926, 248–257, the following interesting theorem: Let χ_1, χ_2, χ_3 be any three primitive Dirichlet characters, i.e. mappings of the multiplicative group of the rational numbers (mod m), for some integer m, into the unit circle in the complex plane. Let

$$L(s, \chi_i)=\sum_{n=1}^{\infty}\frac{\chi_i(n)}{n^s}, \qquad \Re(s)>1$$

be the corresponding Dirichlet L-series.

Theorem 1: *If*

$$Z_3(s)=\prod_{i=1}^{3} L(s, \chi_i), \qquad \Re(s)>1$$

when developed into a Dirichlet series has non-negative coefficients, then

$$(1) \qquad Z_3(s)=\zeta(s)^3$$

or

$$(2) \qquad Z_3(s)=\zeta(s)\zeta_{F_1}(s)$$

or

$$(3) \qquad Z_3(s)=\zeta_{F_2}s),$$

where $\zeta(s)$ *is the Riemann zeta-function,* $\zeta_{F_1}(s)$ *is the Dedekind zeta-function of some quadratic extension of the rational numbers, and* $\zeta_{F_2}(s)$ *is the Dedekind zeta-function of some cubic Abelian extension of the rationals.*

What we propose to prove in the following paper, is that if $\chi_0, \chi_1, \dots, \chi_n$ are any $n+1$ characters (mod m), not necessarily distinct, with at most one of the characters being principal, and if

$$\prod_{j=0}^{n} L(s, \chi_j)$$

has non-negative coefficients, then

$$(4) \qquad \prod_{j=0}^{n} L(s, \chi_j)=\zeta_K(s)$$

where K is a finite Abelian extension of the rationals, and $\zeta_K(s)$ is the corresponding Dedekind zeta-function.

(4) is, unfortunately, only a restricted generalization of Professor Suetuna's result; however, one can see that for a large n there could not possibly be such a simple result as (4).

What we shall do in Section 2 is to reduce the problem to a problem on polynomials in several variables which we will state here.

Theorem 2: *If*

$$f(x_1, x_2, \dots, x_t) = \sum_{j_1=0}^{m_1-1} \dots \sum_{j_t=0}^{m_t-1} a_{j_1,\dots,j_t} x_1^{j_1} \dots x_t^{j_t}$$

is such that

(*a*) $f(0, 0, \dots, 0) = 1$,

(*b*) $a_{j_1,\dots,j_t}$ *are all non-negative rational integers,*

(*c*) *for each i, the greatest common divisor of the set* j_i' *and* m_i *is* 1, *where* j_i' *runs over all* j_i *with at least one* $a_{j_1,\dots,j_i',\dots,j_t} \neq 0$,

(*d*) $f(\zeta_{m_1}^{i_1}, \zeta_{m_2}^{i_2}, \dots, \zeta_{m_t}^{i_t}) \geqq 0$ *for all sets of integers* $i_1, i_2, \dots, i_t$ *where* $\zeta_m = e^{\frac{2\pi i}{m}}$, *then*

$$f(x_1, x_2, \dots, x_t) = \prod_{i=1}^{t} \Big(\sum_{j=0}^{m_i-1} x_i^j \Big) .$$

Theorem 2 will be proved in Section 3 for the case when $t=1$ and all essential details of the proof when $t>1$ will be given in Section 4. Section 1 will consist of a few introductory definitions and lemmas. Section 2 will show the relationship between (4) and Theorem 1, showing that Theorem 2 implies (4).

We may note that (4) will also hold for L-series defined in any algebraic number fields, and the proof is almost identical with the following.

Section 1.

Definitions: $a, b, c, d, h, k, l, m, n, i, j, u,$ will always denote non-negative rational integers. p will always denote a positive rational prime, and

$$\zeta_m = e^{\frac{2\pi i}{m}} .$$

$\varphi(m)$, $\mu(m)$ denote the Euler and the Möbius functions, respectively.

R denotes the rational numbers, and $R(\zeta_m)$ is the field attained by adjoining ζ_m to R.

Lemma 1: The irreducible equation satisfied by ζ_m in R is

$$(5) \qquad g(x) = \prod_{d/m} (1 - x^d)^{\mu(\frac{m}{d})} ,$$

furthermore,

$$(6) \qquad S_{R(\zeta_m), R}(\zeta_m) = \mu(m) ,$$

where $S_{R(\zeta_m), R}(\zeta_m)$ denotes the trace of ζ_m from $R(\zeta_m)$ to R; and

(7) $$(R(\zeta_m):R)=\varphi(m)\,.$$

Proof: The statements in Lemma 1 are all well-known facts about cyclotomic fields and equations and will not be included.

Lemma 2: If α is an algebraic integer contained in an algebraic field F of degree l over R, and if α and all its conjugates over R are positive, then

$$S_{F,R}(\alpha)\geqq l\,.$$

Proof: Denote by $\alpha=\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(l)}$ the conjugates of α. Then by the arithmetic-geometric mean

$$S_{F,R}(\alpha)=\sum_{i=1}^{l}\alpha^{(i)}\geqq l\Big(\prod_{i=1}^{l}\alpha^{(i)}\Big)^{\frac{1}{l}}=l(N_{F,R}(\alpha))^{\frac{1}{l}}\geqq l\,,$$

as $N_{F,R}(\alpha)$ is a positive integer.

Section 2.

Let h denote the least common multiple of all conductors of the characters $\chi_0, \chi_1, \dots, \chi_n$ in the Introduction. Let G be the smallest group of characters defined (mod h) which contains our set (χ_i). Denote by $\tau_1, \tau_2, \dots, \tau_t$ a set of generators of G each of order $m_1, m_2, \dots, m_t$, respectively (i.e. $\tau_i^{m_i}=1$, and m_i is the least positive integer for which this is true).

Denote by $a_{j_1,\dots,j_t}$ the number of times $\tau_1^{j_1}, \tau_2^{j_2}, \dots, \tau_t^{j_t}$ for $j_i=0, 1, \dots, m_i-1$ appears in the set (χ). We see from the definition of the $a_{j_1,\dots,j_t}$ that they are non-negative.

Now for $\Re(s)>1$, we have by the Euler product that

$$L(s,\chi_i)=\prod_p(1-\chi_i(p)p^{-s})^{-1}=\exp\Big\{\sum_{g=1}^{\infty}\sum_p\frac{\chi_i(p^g)}{gp^{gs}}\Big\}\,.$$

Therefore,

$$\prod_{i=0}^{n}L(s,\chi_i)=\prod_{j_1=0}^{m_1-1}\prod_{j_2=0}^{m_2-1}\dots\prod_{j_t=0}^{m_t-1}L(s,\tau_1^{j_1}\dots\tau_t^{j_t})^{a_{j_1,\dots,j_t}}$$

$$=\exp\Big\{\sum_{g=1}^{\infty}\sum_p\sum_{j_1=0}^{m_1-1}\dots\sum_{j_t=0}^{m_t-1}\frac{a_{j_1,\dots,j_t}\tau_1^{j_1}\dots\tau_t^{j_t}(p^g)}{gp^{gs}}\Big\}=\sum_{l=1}^{\infty}\frac{c_l}{l^s}\,.$$

By hypothesis $c_l\geqq 0$, and in particular $c_p\geqq 0$ for all primes p. Now

$$c_p=\sum_{j_1=0}^{m_1-1}\dots\sum_{j_t=0}^{m_t-1}a_{j_1,\dots,j_t}\tau_1^{j_1}\dots\tau_t^{j_t}(p)$$

$$=\sum_{j_1=0}^{m_1-1}\dots\sum_{j_t=0}^{m_t-1}a_{j_1,\dots,j_t}\tau_1^{j_1}(p)\tau_2^{j_2}(p)\dots\tau_t^{j_t}(p)\,.$$

By Dirichlet's theorem regarding primes in an arithmetic progression, we have for every u such that $(u, h)=1$, there exist infinitely many primes p such that

$$p \equiv u \pmod{h} .$$

Hence, for any given t triple $(i_1, \ldots, i_t)$ with $0 \leqq i_j < m_j$, there exists a prime p such that

$$\tau_j(p) = \zeta_{m_j}^{i_j} .$$

Also by Dirichlet's theorem, and the fact that the real point on the line of convergence of a Dirichlet series with positive coefficients is a singularity of the function, we see that $s=1$ is a singularity of $\prod_{i=0}^{n} L(s, \chi_i)$. Hence, at least one character must be principal, and by hypothesis this means only one character is principal, i.e. $a_{0,0,\ldots,0}=1$.

Let $$f(x_1, \ldots, x_t) = \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_t=0}^{m_t-1} a_{j_1,\ldots,j_t} x_1^{j_1} \cdots x_t^{j_t} .$$

Hence, by the above we see $f(x_1, \ldots, x_t)$ satisfies all the conditions of Theorem 2, except perhaps (c). But (c) must be satisfied, otherwise the group of characters G is too large for our purpose.

Assume for the moment we have proved Theorem 2. This would imply that $a_{j_1,\ldots,j_t}=1$ for all $(j_1, \ldots, j_t)$. Therefore, our set of characters (χ) coincides with G. It is then well known by Class Field Theory that there exists an Abelian extension of R whose ray (rayon) group in R will coincide with the kernel of the homomorphisms of G acting on R. We then see that $\prod_{i=0}^{n} L(s, \chi_i)$ must be the zeta-function of this Abelian extension, and so Theorem 2 implies (4).

Section 3.

We shall give here the proof of Theorem 2 when $t=1$.

Case 1. $m_1 = p$. Consider

$$S_{R(\zeta_p), R}(f(\zeta_p)) = S_{R(\zeta_p), R}\Big(\sum_{j=0}^{p-1} a_j \zeta_p^j\Big) = \sum_{j=0}^{p-1} a_j S_{R(\zeta_p), R}(\zeta_p^j)$$
$$= (p-1)a_0 - \Big(\sum_{j=1}^{p-1} a_j\Big) = (p-1) - \Big(\sum_{j=1}^{p-1} a_j\Big) < p-1 ,$$

by properties (a), (b) and (c) of Theorem 2. By assumption (d), $f(\zeta_p)$ is a totally positive ($\geqq 0$) algebraic integer in $R(\zeta_p)$ which is of degree $p-1$ over R. Hence, by Lemma 2

$$f(\zeta_p) = 0 .$$

Therefore, by Lemma 1,

$$f(x) = \sum_{j=0}^{p-1} x^j .$$

Case 2. $w(m) = h > 1$ where if $m = m_1 = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}$, $c_i > 1$, then $w(m) = c_1 + c_2 + \cdots + c_r$.

We have shown that Theorem 2 is true if $w(m)=1$. Proceeding by induction, assume the theorem is true if $w(m)\leqq h-1$.

Define

$$g_n(y)=\sum_{j=0}^{n-1} a_{pj}y^j$$

where $p=p_1$ and $n=m/p$. Then

$$(8)\qquad \frac{1}{p}\sum_{i=0}^{p-1} f(\zeta_m^{v+ni})=\frac{1}{p}\sum_{i=0}^{p-1}\sum_{j=0}^{m-1} a_j\zeta_m^{j(v+ni)}=\sum_{j=0}^{m-1} a_j\zeta_m^{vj}\Big(\frac{1}{p}\sum_{i=0}^{p-1}\zeta_m^{jni}\Big)$$

$$=\sum_{j=0}^{n-1} a_{pj}\zeta_m^{vpj}=\sum_{j=0}^{n-1} a_{pj}\zeta_n^{vj}=g_n(\zeta_n^v)\,.$$

So by (8), we see that $g_n(y)$ is non-negative for $y=\zeta_n^v$ for $v=0, 1, 2, \ldots, n-1$. By the definition of $g_n(y)$ we see that the coefficients are non-negative and that $g_n(0)=1$. Hence, $g_n(y)$ satisfies every condition of Theorem 2, with $t=1$ and m replaced by n, except perhaps condition (c).

Let d be the greatest common divisor of the j' and n where j' runs over all j such that $a_{pj'}\neq 0$. Hence,

$$g_n(y)=\sum_{j=0}^{n/d-1} a_{pdj}y^{dj}\,.$$

If

$$\bar{g}_n(z)=\sum_{j=0}^{n/d-1} a_{pdj}z^j\,,\quad y^d=z\,,$$

then $\bar{g}_n(z)$ satisfies all the conditions of Theorem 2 with m replaced by n/d. So by induction,

$$\bar{g}_n(z)=\sum_{j=0}^{n/d-1} z^j$$

or

$$(9)\qquad g_n(x^p)=\sum_{j=0}^{n/d-1} x^{pdj}\,.$$

By (9) the roots of $g_n(x^p)$ are ζ_m^l where l is not divisible by m/pd. Furthermore, by (9) $g_n(x^p)$ is non-negative for any mth root of unity.

As the elements on the left-hand side of (8) are non-negative, we have that $f(\zeta_m^l)=0$ if $m/pd\nmid l$. Hence, $g_n(x^p)$ divides $f(x)$, or

$$(10)\qquad f(x)=g_n(x^p)h(x)$$

where the degree of $h(x)$ is $<pd$ by (9). Also by (9) and the fact that $f(x)$ has non-negative coefficients, $h(x)$ has non-negative coefficients:

$$h(0)=\frac{f(0)}{g_n(0)}=1\,.$$

Again by (9) $g_n(\zeta_{pd}^{pi})>0$, so $h(\zeta_{pd}^i)\geqq 0$ for all i.

If $w(pd)<w(m)$, then $h(x)=\dfrac{1-x^{pd}}{1-x}$, so by (9) and (10),

$$f(x)=\frac{1-x^m}{1-x} .$$

If $w(pd)=w(m)$, then as $d\mid n$, $pn=m$, we have $d=n$. Hence by (9), $g_n(x^p)=1$.

In formula (8) let $v=0$, so

$$\frac{1}{p}\sum_{i=0}^{p-1} f(\zeta_m^{ni})=1 ,$$

or

$$\sum_{i=1}^{p-1} f(\zeta_p^i)=p-f(1) . \tag{11}$$

Now $f(1)\geqq 2$, as the coefficients of $f(x)$ are non-negative, $f(x)$ satisfies condition (c), and $f(0)=1$.

So by (11)

$$S_{R(\zeta_p),R}(f(\zeta_p))<p-1 ,$$

or by Lemma 2

$$S_{R(\zeta_p),R}(f(\zeta_p))=0 .$$

Hence, by (11)

$$f(1)=p . \tag{12}$$

(12) would give a contradiction if m had two different prime factors, as then $f(1)=p_1$, $f(1)=p_2$ with $p_1\neq p_2$.

So we are reduced to the case $m=p^c$, $c>1$.

Again by (8), letting $v=kp^{c-2}$,

$$S_{R(\zeta_{p^2}),R}(f(\zeta_{p^2}))=\sum_{(i,p)=1} f(\zeta_{p^2}^i)=\sum_{k=1}^{p-1}\sum_{l=0}^{p-1} f(\zeta_{p^2}^{k+lp}) \tag{13}$$

$$=\sum_{k=1}^{p-1}\sum_{l=0}^{p-1} f(\zeta_{p^c}^{kp^{c-2}+lp^{c-1}})=\sum_{k=1}^{p-1} p=p^2-p .$$

But by Lemma 2, i.e. the arithmetic-geometric inequality, this implies $f(\zeta_{p^2}^i)=1$ for $(i,p)=1$. Similarly, we see that $f(\zeta_{p^c}^i)=1$ for $(i,p)=1$. So

$$f(\zeta_{p^c}^i)=\begin{cases} 1 & \text{if } p^{c-1}\nmid i \\ p & \text{if } i=0 \\ 0 & \text{otherwise.}\end{cases} \tag{14}$$

Now compare $f(x)$ with the polynomial

$$F(x)=1-p^{1-c}\sum_{j=0}^{p^{c-1}-1} x^{pj}+p^{1-c}\sum_{j=0}^{p^c-1} x^j .$$

We note that $F(x)$ has the identical behavior as $f(x)$ at the p^c points in (14). As the degrees of $f(x)$ and $F(x)$ are both less than p^c, we must have that $f(x)$ and $F(x)$ are identically equal. As $c\geqq 2$, we see that $F(x)$ will have non-integral coefficients, which gives a contradiction.

Section 4.

In this section we shall prove Theorem 2 for $t=2$, and note that this proof can be carried over automatically for $t>2$:

$$f(x_1, x_2)=\sum_{j_1=0}^{m_1-1}\sum_{j_2=0}^{m_2-1} a_{j_1,j_2}x_1^{j_1}x_2^{j_2}.$$

Assume Theorem 2 is true if $w(m_1m_2)\leqq h$. We note that we have proved the case when $h=1$, as then m_1 or m_2 equals 1 and this falls under the case when $t=1$. Assume $w(m_1m_2)=h+1$ and let $p|m_1$, $n_1=m_1/p$. Let

$$f(x_1, x_2)=g_{0,p}(x_1^p, x_2)+g_{1,p}(x_1, x_2)$$

where

$$g_{0,p}(x_1^p, x_2)=\sum_{j_1=0}^{n_1-1}\sum_{j_2=0}^{m_2-1} a_{pj_1,j_2}x_1^{pj_1}x_2^{j_2}$$

and $g_{1,p}(x_1, x_2)$ contains the other terms of $f(x_1, x_2)$. Denote $x_1^p=\bar{x}_1$. So $g_{0,p}(\bar{x}_1, x_2)$ is a polynomial of degree $<n_1$ in $\bar{x}_1$, of degree $<m_2$ in x_2 and

$$g_{0,p}(\zeta_{n_1}^{v_1}, \zeta_{m_2}^{v_2})=\frac{1}{p}\sum_{i=0}^{p-1}f(\zeta_{m_1}^{v_1+n_1i}, \zeta_{m_2}^{v_2}).$$

Hence, $g_{0,p}(\bar{x}_1, x_2)$ satisfies every condition of Theorem 2, except perhaps condition (c), with m_1, m_2 replaced by n_1, m_2.

As $w(n_1m_2)<w(m_1m_2)$ we have by induction that (concerning d_1, d_2, see d in Section 3)

$$g_{0,p}(x_1^p, x_2)=\sum_{j_1=0}^{n_1/d_1-1}\sum_{j_2=0}^{m_2/d_2-1} x_1{}^{pd_1j_1}x_2{}^{d_2j_2}.$$

Unless $d_1=n_1$, $d_2=m_2$, as in Section 3 then

$$f(x_1, x_2)=\sum_{j_1=0}^{m_1-1}\sum_{j_2=0}^{m_2-1} x_1^{j_1}x_2^{j_2}.$$

If $d_1=n_1$, $d_2=m_2$, then

$$\sum_{i=0}^{p-1}f(\zeta_{m_1}^{v_1+n_1i}, \zeta_{m_2}^{v_2})=p \tag{15}$$

for all v_1, v_2. Then (15) yields a contradiction.

The case when $t>2$, proceeds precisely as in the case when $t=2$.