117. Probability-theoretic Investigations on Inheritance. XV_3. Detection of Interchange of Infants.

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5. Main results.

After a preliminary preparation of the preceding section we shall now enter our main discourse. Here it will be convenient, contrary to the agreement in the previous section, to take the order of two members of a mating into account. Since it will be the same whether we classify the types according to a father or to a mother of a mating, we shall prefer the latter.

We now denote by

$$(5.1) \qquad G(ij, hk) \qquad (i, j, h, k=1, \cdots, m)$$

the probability of an event that a triple consisting of a mother A_{ij}, *a father* A_{hk} *and an apparent child is presented and the detection of interchange is possible against another triple*; an agreement corresponding to the one immediately subsequent to (3.3) of IV is made here again. The probability $G(ij, hk)$ consists of two parts; the one corresponds to case where the detection of interchange is *possible indifferent to another triple*, and the other to case where it becomes *possible only by taking another triple into account*. These two partial probabilities be denoted by

$$(5.2) \qquad G_0(ij, hk), \qquad \Phi(ij, hk),$$

respectively, the sum being

$$(5.3) \qquad G(ij, hk)=G_0(ij, hk)+\Phi(ij, hk).$$

The symmetry properties will be evident:

$$(5.4) \quad G_0(ij, hk)=G_0(hk, ij), \quad \Phi(ij, hk)=\Phi(hk, ij); \quad G(ij, hk)=G(hk, ij).$$

Now, if the first mating can produce only one type of child, then the interchange is detectable by means of the first triple alone, provided it is detectable at any rate. Hence, we get

$$(5.5) \qquad G(ii, ii)=G_0(ii, ii), \qquad \Phi(ii, ii)=0,$$

$$(5.6) \qquad G(ii, hh)=G_0(ii, hh), \qquad \Phi(ii, hh)=0 \qquad (h\neq i).$$

Since the mating $A_{ii}\times A_{ii}$ can produce A_{ii} alone, an apparent child other than A_{ii} is detectable and hence

$$(5.7) \qquad G(ii, ii)=G_0(ii, ii)=\bar{A}_{ii}^2(1-\bar{A}_{ii})=p_i^4(1-p_i^2).$$

Similarly, by remembering that the order within a mating is to be taken into account, we get

(5.8) $G(ii, hh)=G_0(ii, hh)=\overline{A}_{ii}\overline{A}_{hh}(1-\overline{A}_{ih})=p_i^2p_h^2(1-2p_ip_h) \quad (h\neq i).$

Next, a mating cosisting of a mother A_{ii} and a father A_{ih} $(h\neq i)$ can produce A_{ii} and A_{ih}. Hence, we get

(5.9) $G_0(ii, ih)=\overline{A}_{ii}\overline{A}_{ih}(1-\overline{A}_{ii}-\overline{A}_{ih})=2p_i^3p_h(1-p_i^2-2p_ip_h) \quad (h\neq i).$

If, for this mating, an apparent child of the second mating, a true child of the first mating, is A_{ii} and the second mating can produce A_{ih} but not A_{ii}, then the detection is possible even when an apparent child of the first mating is A_{ih}. Similarly, if an apparent child of the second mating is A_{ih} and the second mating can produce A_{ii} but not A_{ih}, then the detection is possible even when an apparent child of the first mating is A_{ii}. Since the mating $A_{ii}\times A_{ih}$ produces A_{ii} and A_{ih} with equal probability 1/2, we thus get

(5.10)
$$\begin{aligned}\Phi(ii, ih)&=\overline{A}_{ii}\overline{A}_{ih}(\tfrac{1}{2}\varphi(-ii, +ih)+\tfrac{1}{2}\varphi(-ih, +ii))\\&=2p_i^3p_h(p_ip_h(1-p_i)+\tfrac{1}{2}p_i^2(1-p_h)^2) \qquad (h\neq i),\end{aligned}$$

whence follows, together with (5.9),

(5.11) $G(ii, ih)=p_i^3p_h(2(1+p_i)(1-p_i-p_ip_h)+p_i^2(1-p_h)^2) \quad (h\neq i).$

The mating consisting of a mother A_{ii} and a father $A_{hk}(h, k\neq i; h\neq k)$ can produce A_{ih} and A_{ik} equally probably. Hence, we obtain

(5.12)
$$G_0(ii, hk)=\overline{A}_{ii}\overline{A}_{hk}(1-\overline{A}_{ih}-\overline{A}_{ik})=2p_i^2p_hp_k(1-2p_i(p_h+p_k)) \qquad (h, k\neq i; h\neq k).$$

If, for this mating, an apparent child of the second mating is A_{ih} and the second mating can produce A_{ik} but not A_{ih}, then the detection is possible even when an apparent child of the first mating is A_{ik}. The same is true if we commute the suffices h and k. Hence, we get

(5.13)
$$\begin{aligned}\Phi(ii, hk)&=\overline{A}_{ii}\overline{A}_{hk}(\tfrac{1}{2}\varphi(-ih, +ik)+\tfrac{1}{2}\varphi(-ik, +ih))\\&=2p_i^3p_hp_k(p_h+p_k-2(1+p_i)p_hp_k) \qquad (h, k\neq i; h\neq k),\end{aligned}$$

whence follows, together with (5.12),

(5.14)
$$G(ii, hk)=2p_i^2p_hp_k(1-p_i(p_h+p_k)-2(1+p_i)p_ip_hp_k) \qquad (h, k\neq i; h\neq k).$$

The cases of homozygotic mother of the first mating have thus essentially been worked out.

In view of symmetry property (5.4), we get, from (5.9) to (5.11),

(5.15) $G_0(ij, ii)=2p_i^3p_j(1-p_i^2-2p_ip_j) \quad (i\neq j),$

(5.16) $\Phi(ij, ii)=2p_i^3p_j(p_ip_j(1-p_i)+\tfrac{1}{2}p_i^2(1-p_j)^2) \quad (i\neq j),$

(5.17) $G(ij, ii)=p_ip_j(2(1+p_i)(1-p_i-p_ip_j)+p_i^2(1-p_j)^2) \quad (i\neq j),$

Similarly, the case $A_{ij}\times A_{hh}$ $(i\neq j; h\neq i, j)$ has essentially been treated, as $A_{ii}\times A_{hk}$, in (5.12) to (5.14).

The mating $A_{ij}\times A_{ij}$ produces A_{ii}, A_{jj} and A_{ij} with probabilities

1/4, 1/4 and 1/2, respectively. Hence, we get

$$(5.18)\quad G_0(ij,\ ij)=\overline{A}_{ij}^2(1-\overline{A}_{ii}-\overline{A}_{jj}-\overline{A}_{ij})=4p_i^2p_j^2(1-(p_i+p_j)^2)\quad (i\neq j).$$

If an apparent child of the second mating is A_{ii} and the second mating cannot produce A_{ii}, then the interchange is detectable even when an apparent child of the first mating is A_{jj} or A_{ij}. The same is valid also if the suffices i and j are commuted. Further, if an apparent child of the second mating is A_{ij} and the second mating cannot produce A_{ij}, then the detection is possible when an apparent child of the first mating is A_{ii} or A_{jj}. Thus, we get

$$(5.19)\quad \begin{aligned}\Phi(ij,\ ij)=\overline{A}_{ij}^2(&\tfrac{1}{4}\varphi(-ii,\ +jj+ij)+\tfrac{1}{4}\varphi(-jj,\ +ii+ij)\\ &+\tfrac{1}{2}\varphi(-ij,\ +ii+jj))\\ =p_i^2p_j^2(&3(p_i^2+p_j^2)+4p_ip_j-6p_ip_j(p_i+p_j)+2p_i^2p_j^2)\quad (i\neq j),\end{aligned}$$

whence follows, together with (5.18),

$$(5.20)\quad G(ij,\ ij)=p_i^2p_j^2(4-(p_i^2+p_j^2)-4p_ip_j-6p_ip_j(p_i+p_j)+2p_i^2p_j^2)\quad (i\neq j).$$

The mating $A_{ij}\times A_{ih}(i\neq j;h\neq i,j)$ produces A_{ii}, A_{ij}, A_{ih} and A_{jh} equally probably. We get

$$(5.21)\quad \begin{aligned}G_0(ij,\ ih)&=\overline{A}_{ij}\overline{A}_{ih}(1-\overline{A}_{ii}-\overline{A}_{ij}-\overline{A}_{ih}-\overline{A}_{jh})\\ &=4p_i^2p_jp_h(1-p_i^2-2p_i(p_j+p_h)-2p_jp_h)\quad (i\neq j;h\neq i,j).\end{aligned}$$

If an apparent child of the second mating is A_{ii} which is incompatible, then the detection is possible even when an apparent child of the first mating is A_{ij}, A_{ih} or A_{jh}. The same is valid with corresponding modification for other three types of an apparent chlid of the second mating. Thus, we get

$$(5.22)\quad \begin{aligned}\Phi(ij,\ ih)=\overline{A}_{ij}\overline{A}_{ih}(&\tfrac{1}{4}\varphi(-ii,\ +ij+ih+jh)+\tfrac{1}{4}\varphi(-ij,\ +ii+ih+jh)\\ &+\tfrac{1}{4}\varphi(-ih,\ +ii+ij+jh)+\tfrac{1}{4}\varphi(-jh,\ +ii+ij+ih))\\ =p_i^2p_jp_h(&3p_i^2+2p_i(3-2p_i)(p_j+p_h)+p_i^2(p_j^2+p_h^2)\\ &+2(3-6p_i-4p_i^2)p_jp_h-4p_ip_jp_h(p_j+p_h))\\ &\qquad (i\neq j;h\neq i,j),\end{aligned}$$

whence follows, together with (5.21),

$$(5.23)\quad \begin{aligned}G(ij,\ ih)=p_i^2p_jp_h(&4-p_i^2-2p_i(1+2p_i)(p_j+p_h)+p_i^2(p_j^2+p_h^2)\\ &-2(1+6p_i+4p_i^2)p_jp_h-4p_ip_jp_h(p_j+p_h))\\ &\qquad (i\neq j;\ h\neq i,j).\end{aligned}$$

Finally, the mating $A_{ij}\times A_{hk}(i\neq j,\ h\neq k;\ h,\ k\neq i,j)$ produces A_{ih} A_{jh}, A_{ik} and A_{jk} equally probably. We get

$$(5.24)\quad \begin{aligned}G_0(ij,\ hk)&=\overline{A}_{ij}\overline{A}_{hk}(1-\overline{A}_{ih}-\overline{A}_{jh}-\overline{A}_{ik}-\overline{A}_{jk})\\ &=4p_ip_jp_hp_k(1-2(p_i+p_j)(p_h+p_k))\\ &\qquad (i\neq j;\ h\neq k;\ h,\ k\neq i,j).\end{aligned}$$

Similarly as above, we further get

$$\begin{aligned}\Phi(ij, hk) &= \bar{A}_{ij}\bar{A}_{hk}(\tfrac{1}{4}\varphi(-ih, +jh+ik+jk)+\tfrac{1}{4}\varphi(-jh, +ih+ik+jk)\\ &\quad +\tfrac{1}{4}\varphi(-ik, +ih+jh+jk)+\tfrac{1}{4}\varphi(-jk, +ih+jh+ik))\\ &= 2p_ip_jp_hp_k(3(p_i+p_j)(p_h+p_k)-2p_ip_j(p_h+p_k)\\ &\quad -2p_hp_k(p_i+p_j)-2p_ip_j(p_h^2+p_k^2)-2p_hp_k(p_i^2+p_j^2)\\ &\quad -8p_ip_jp_hp_k) \qquad (i\neq j;\ h\neq k,\ h, k\neq i, j),\end{aligned} \tag{5.25}$$

whence follows, together with (5.24),

$$\begin{aligned}G(ij, hk) &= 2p_ip_jp_hp_k(2-(p_i+p_j)(p_h+p_k)-2p_ip_j(p_h+p_k)\\ &\quad -2p_hp_k(p_i+p_j)-2p_ip_j(p_h^2+p_k^2)-2p_hp_k(p_i^2+p_j^2)\\ &\quad -8p_ip_jp_hp_k) \qquad (i\neq j;\ h\neq k;\ h, k\neq i, j).\end{aligned} \tag{5.26}$$

Thus, all the cases have essentially been exhausted. By summing up the partial probabilities over all possible cases, we shall obtain the whole probability of detecting the interchange of infants. However, we shall here compute more precisely the partial probabilities by summing up the G_0's and Φ's in (5.2) with a fixed type of mother over all possible types of father. We thus get

$$\begin{aligned}G_0(ii) &\equiv G_0(ii, ii)+\sum_{h\neq i}(G_0(ii, hh)+G_0(ii, ih))+\sum_{h,k\neq i}{}' G_0(ii, hk)\\ &= p_i^2(1-2(2S_2-S_3)p_i+2p_i^3-p_i^4),\end{aligned} \tag{5.27}$$

$$\begin{aligned}G_0(ij) &\equiv G_0(ij, ii)+G_0(ij, jj)+G_0(ij, ij)\\ &\quad +\sum_{h\neq i,j}(G_0(ij, ih)+G_0(ij, jh)+G_0(ij, hh))+\sum_{h,k\neq i,j}{}' G_0(ij, hk)\\ &= 2p_ip_j(1-2(2S_2-S_3)(p_i+p_j)+2(p_i^3+p_j^3)\\ &\quad -(p_i^4+p_j^4)+4p_i^2p_j^2) \qquad (i\neq j),\end{aligned} \tag{5.28}$$

$$\begin{aligned}\Phi(ii) &\equiv \Phi(ii, ii)+\sum_{h\neq i}(\Phi(ii, hh)+\Phi(ii, ih))+\sum_{h,k\neq i}{}' \Phi(ii, hk)\\ &= p_i^2(2(S_2-S_3-S_2^2+S_4)-2(S_2^2-S_4)p_i-(1-S_3)p_i^2\\ &\quad +(1+4S_2)p_i^3+5p_i^5),\end{aligned} \tag{5.29}$$

$$\begin{aligned}\Phi(ij) &= \Phi(ij, ii)+\Phi(ij, jj)+\Phi(ij, ij)\\ &\quad +\sum_{h\neq i,j}(\Phi(ij, ih)+\Phi(ij, jh)+\Phi(ij, hh))+\sum_{h,k\neq i,j}{}' \Phi(ij, hk)\\ &= p_ip_j(2(3S_2-2S_3-S_2^2+S_4)(p_i+p_j)-2(S_2^2-S_4)(p_i^2+p_j^2)\\ &\quad -4(S_2+S_3+2S_2^2-2S_4)p_ip_j-(3-S_3)(p_i^3+p_j^3)\\ &\quad -4S_2p_ip_j(p_i+p_j)+2(1+2S_2)(p_i^4+p_j^4)\\ &\quad +8S_2p_ip_j(p_i^2+p_j^2)-8p_i^2p_j^2+8p_ip_j(p_i^3+p_j^3)\\ &\quad +7p_i^2p_j^2(p_i+p_j)-5(p_i^6+p_j^6)-8p_ip_j(p_i^4+p_j^4)\\ &\quad -2p_i^3p_j^3) \qquad (i\neq j).\end{aligned} \tag{5.30}$$

The sum of (5.27) and (5.29), and of (5.28) and (5.30) become

$$\begin{aligned}G(ii) &= p_i^2(1-2(S_2+S_2^2-S_4)p_i-2(S_2^2-S_4)p_i^2\\ &\quad +(1+S_3)p_i^3+4S_2p_i^4-5p_i^6),\end{aligned} \tag{5.31}$$

$$\begin{aligned}G(ij) &= p_ip_j(2-2(S_2+S_2^2-S_4)(p_i+p_j)-2(S_2^2-S_4)(p_i^2+p_j^2)\\ &\quad -4(S_2+S_3+2S_2^2-2S_4)p_ip_j+(1+S_3)(p_i^3+p_j^3)\\ &\quad -4S_2p_ip_j(p_i+p_j)+4S_2(p_i^4+p_j^4)+8S_2p_ip_j(p_i^2+p_j^2)\\ &\quad +8p_ip_j(p_i^3+p_j^3)+7p_i^2p_j^2(p_i+p_j)-5(p_i^6+p_j^6)\\ &\quad -8p_ip_j(p_i^4+p_j^4)-2p_i^3p_j^3) \qquad (i\neq j),\end{aligned} \tag{5.32}$$

respectively. On the other hand, we get by summation

$$(5.33)\quad \sum_{i=1}^{m} G_0(ii)=S_2-4S_2S_3+2S_5+2S_3^2-S_6,$$

$$(5.34)\quad \sum_{i,j}' G_0(ij)=1-S_2-8S_2^2+4S_4+12S_2S_3-6S_5-2S_6;$$

$$(5.35)\quad \begin{aligned}\sum_{i=1}^{m} \Phi(ii)=&2S_2S_3-S_5-2S_3^2+S_6-2S_2^2S_3+2S_3S_4\\&-2S_2^2S_4+2S_4^2+S_3S_5+4S_2S_6-5S_8,\end{aligned}$$

$$(5.36)\quad \begin{aligned}\sum_{i,j}' \Phi(ij)=&6S_2^2-3S_4-10S_2S_3+5S_5-4S_2^3+4S_2S_4+2S_6\\&-6S_2^2S_3+10S_3S_4+16S_2S_5-20S_7-4S_2^4\\&+18S_2^2S_4-7S_4^2-S_3S_5-20S_2S_6+14S_8.\end{aligned}$$

Further, summing up (5.31), (5.32) over all possible suffices, or rather adding (5.33) and (5.35), (5.34) and (5.36), we get

$$(5.37)\quad \begin{aligned}\sum_{i=1}^{m} G(ii)=&S_2-2S_2S_3+S_5-2S_2^2S_3+2S_3S_4\\&-2S_2^2S_4+2S_4^2+S_3S_5+4S_2S_6-5S_8,\end{aligned}$$

$$(5.38)\quad \begin{aligned}\sum_{i,j}' G(ij)=&1-S_2-2S_2^2+S_4+2S_2S_3-S_5-4S_2^3+4S_2S_4\\&-6S_2^2S_3+10S_3S_4+16S_2S_5-20S_7-4S_2^4\\&+18S_2^2S_4-7S_4^2-S_3S_5-20S_2S_6+14S_8.\end{aligned}$$

These quantities represent the *partial probabilities* of detecting the interchange of infants for homo- and heterozygotic mothers of the first mating, respectively.

On the other hand, the sums of (5.33) and (5.34), and of (5.35) and (5.36) become respectively

$$(5.39)\quad G_0=1-8S_2^2+4S_4+8S_2S_3-4S_5+2S_3^2-3S_6,$$

$$(5.40)\quad \begin{aligned}\Phi=&6S_2^2-3S_4-8S_2S_3+4S_5-4S_2^3-2S_3^2+4S_2S_4+3S_6\\&-8S_2^2S_3+12S_3S_4+16S_2S_5-20S_7-4S_2^4+16S_2^2S_4\\&-5S_4^2-16S_2S_6+9S_8.\end{aligned}$$

These quantities represent the partial probabilities of detecting the interchange respectively without or only by taking the second triple into account. The sum of the last two quantities or, what is the same thing, the sum of (5.37) and (5.38) yields ultimately the *whole probabibity of detcting the interchange of infants*

$$(5.41)\quad \begin{aligned}G\equiv G_0+\Phi=&1-2S_2^2+S_4-4S_2^3+4S_2S_4-8S_2^2S_3+12S_3S_4\\&+16S_2S_5-20S_7-4S_2^4+16S_2^2S_4\\&-5S_4^2-16S_2S_6+9S_8.\end{aligned}$$

6. Additional remarks.

We have hitherto discussed the problem by supposing that the infants are actually interchanged between two matings. However, as already stated at the end of § 1, this supposition is essentially indifferent. The whole probability obtained in (5.41) may thus be regarded as the one that the true matings for two infants can be determined.

We shall make here a further notice concerning a relation between partial probabilities G_0 and Φ given in (5.39) and (5.40). We have derived G_0 according to the first triple. But, the same is also valid according to the second triple. Namely, the quantity $G_0(ij, hk)$ defined in (5.2) may be regarded as the probability of an event that the second mating $A_{ij}\times A_{hk}$ appears and then the detection of interchange is possible indifferent to the first triple. On the other hand, the quantity $\Phi(ij, hk)$ also defined in (5.2) is the probability of an event that, among the cases where the second mating cannot produce its apparent infant, the detection is impossible by taking only the first triple with the mating $A_{ij}\times A_{hk}$ into account. Therefore, every term constituting $\Phi(ij, hk)$ for every (ij, hk) is contained in the sum (5.39) without repetition. Thus, we concluded that *an inequality*

(6.1) $$G_0\geqq\Phi,\quad \text{i.e.}\quad G_0\geqq\tfrac{1}{2}G\geqq\Phi$$

holds good. In particular, the probability G_0 of cases where the detection is possible without referring the second triple is not less than the half of the whole probability $G=G_0+\Phi$.

The difference $G_0-\Phi$ can be really explained as the probability of an event that the detection is *possible within every triple.*

For instance, in a special case of MN blood type, we get

(6.2) $$G_{0MN}=2st-5s^2t^2+6s^3t^3,$$

(6.3) $$\Phi_{MN}=2st-9s^2t^2+14s^3t^3-2s^4t^4.$$

The difference is surely non-negative:

(6.4) $$G_{0MN}-\Phi_{MN}=2s^2t^2(2-4st+s^2t^2)\geqq 0,$$

since $st\leqq 1/4$. Moreover, the equality sign in the last inequality may be rejected unless a trivial case $st=0$ occurs.

—To be continued—