123. Simple Proof of a Theorem of Ankeny on Dirichlet Series.

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Mr. Ankeny has proved recently the following theorem[1]:

Let $\chi_1, \chi_2, \cdots, \chi_n$ be n primitive Dirichlet characters with at most one of them being principal, and let $L(s; \chi_i)$ be Dirichlet L-series corresponding to χ_i. If then all coefficients of the Dirichlet series

$$Z(s)=\prod_{i=1}^{n} L(s; \chi_i)$$

are non-negative, then $Z(s)$ is the Dedekind ζ-function of some Abelian extension of the rational number field.

In the following I shall give a simple proof of this interesting theorem. Let H be the set of all distinct characters among $\chi_1\ \chi_2, \cdots, \chi_n$, and G the group of characters generated by H. Further let a_χ be the number of times χ, one of the elements of G, appears in $\{\chi_1, \chi_2, \cdots, \chi_n\}$ and f the least common multiple of all the conducters of our characters. It is easily shown that

$$\begin{aligned} Z(\mathrm{s}) &= \prod_{\chi\in H} L(s; \chi)^{a_\chi} \\ &= \exp\Big(\sum_{p}\sum_{g=1}^{\infty}\big(\sum_{\chi\in H} a_\chi \chi(p^g)/gp^{gs}\big)\Big). \end{aligned}$$

So we get as the coefficient of $1/p^s$ in $Z(s)$

$$\sum_{\chi\in H} a_\chi\chi(p)=\sum_{\chi\in G} a_\chi\chi(p).$$

By the hypothesis of our theorem

$$\sum_{\chi\in G} a_\chi\chi(p)\geqq 0 \qquad \text{for all prime numbers } p,$$

so that we have by Dirichlet's prime number theorem

$$F(u)=\sum_{\chi\in G} a_\chi\chi(u)\geqq 0 \tag{1}$$

for each u of the representatives of the reduced classes mod f.

From (1) we get

$$ga_\chi=\sum_{u \bmod f} \chi^{-1}(u)F(u), \tag{2}$$

where g is the order of G; in particular

$$ga_{\chi_0}=\sum_{u \bmod f} F(u), \qquad (u, f)=1,$$

where χ_0 denotes the principal character. From (1) and (2) follows

$$\begin{aligned} ga_\chi &\leqq \sum_{u \bmod f} |\chi^{-1}(u)F(u)| = \sum_{u \bmod f} F(u), \quad (u, f)=1, \\ &= ga_{\chi_0}, \\ a_\chi &\leqq a_{\chi_0}. \end{aligned}$$

By this inequality we can see $a_{\chi_0} \geqq 1$, because $a_\chi > 0$ for some character χ. But by the hypothesis $a_{\chi_0} \leqq 1$. We have therefore

$$a_{\chi_0} = 1, \tag{3}$$

and $a_\chi = 1$ or 0 for all $\chi \in G$. From (2) follows

$$\begin{aligned} ga_{\chi^{-1}} &= \sum_{u \bmod f} \chi(u)F(u) \\ &= \sum_{u \bmod f} \overline{\chi(u)}\,\overline{F(u)} = \sum_{u \bmod f} \chi^{-1}(u)F(u) \\ &= ga_\chi, \end{aligned}$$

so that

$$a_{\chi^{-1}} = a_\chi . \tag{4}$$

If $a_\chi = 1$, then

$$\begin{aligned} 0 &= ga_{\chi_0} - ga_\chi \\ &= \sum_{u \bmod f} (1 - \Re\chi(u))F(u), \quad (u, f)=1. \end{aligned}$$

Thus we see that if $a_\chi = 1$, $F(u) \neq 0$ with $(u, f)=1$, then $\chi(u)=1$. Accordingly if $a_\chi = 1$ and $a_{\chi'} = 1$, then from $F(u) \neq 0$ with $(u, f)=1$ follows $\chi(u)=1$, $\chi'(u)=1$, and also $\chi\chi'(u)=1$. So we get

$$\begin{aligned} ga_{\chi\chi'} &= \sum_{u \bmod f} (\chi\chi'(u))^{-1}F(u) \\ &= \sum_{\substack{u \bmod f \\ F(u) \neq 0}} F(u) = \sum_{u \bmod f} F(u), \quad (u, f)=1, \\ &= ga_{\chi_0}; \end{aligned}$$

$$a_{\chi\chi'} = a_{\chi_0} = 1 \quad \text{for} \quad a_\chi = a_{\chi'} = 1. \tag{5}$$

By (3), (4) and (5) H is a group and consequently coincides with G. Therefore $a_\chi = 1$ for all χ in G, and we see $Z(s)$ is the Dedekind ζ-function defined in the field corresponding to the character group G.

Remarks.

Ankeny's auxiliary theorem (theorem 2) is not correct, if $t > 1$. For, in the case $t=2$, $m_1 = m_2 = 3$,

$$f(x_1, x_2) = 1 + x_1x_2 + x_1^2x_2^2$$

satisfies all the conditions of this theorem; but $f(x_1, x_2)$ is not equal to

$$(1 + x_1 + x_1^2)(1 + x_2 + x_2^2).$$

If now we consider the function

$$L(s;\chi_0)^2\ L(s;\chi)\ L(s;\bar{\chi}),$$

where χ is a primitive character of order >4, its coefficients are all non-negative, but it can not be represented by the product of Dedekind ζ-functions. Observing such an example, it seems difficult, as Ankeny remarks, to get a result in general, when the principal character appears more than one time[2].

References.

1) Ankeny, N. C.: A generalization of a theorem of Suetuna on Dirichlet series. Proc. Japan Acad., **28**, 389-395 (1952).

2) Suetuna, Z.: Bemerkung über das Produkt von L-Funktionen. Tōhoku Math. Journal, **27**, 248-257 (1926).