24. *On the Existence of Solutions of a System of Quadratic Equations and Its Geometrical Application*

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S.S. Chern and N.H. Kuiper recently proved a theorem in a paper[1] which is contained in the following theorem as the cases $q=2, 3$.

Theorem 1. *Let M be a compact Riemannian manifold with the property that at every point there is a q-dimensional linear subspace in the tangent space along whose plane elements the sectional curvatures are non-positive. Then M cannot be isometrically imbedded in an Euclidean space of dimension $n+q-1$.*

According to their argument, in order to prove the above theorem, it is sufficient to prove an algebraic theorem as follows:

Theorem 2. *Let*

$$\Psi_\alpha(x) \equiv A_{\alpha ij}\, x^i x^j = 0,^{2)}$$

$$A_{\alpha ij} = A_{\alpha ji};\ i, j = 1, 2, \dots, n;\ \alpha = 1, 2, \dots, N \qquad (1)$$

be a system of quadratic equations in x^i. If

$$L(x, y) \equiv \sum_{\alpha=1}^{N} (A_{\alpha ih}A_{\alpha jk} - A_{\alpha ik}A_{\alpha jh})x^i y^j x^h y^k \leqq 0 \qquad (2)$$

for any x^i, y^j, it has a non-trivial real solution in x^i, when $N<n$.

S.S. Chern and N.H. Kuiper stated the theorem as a probable conjecture[3] and proved only the cases $n=2, 3$ separately by means of an algebraic method. We shall give a simple proof of Theorem 2 by means of an analytical method.

Proof of Theorem 2. Let us consider $x^1, \dots, x^n$ as the orthonormal coordinates of a point x in an Euclidean space of dimension n. Let be $H(x)=\sum_{\alpha=1}^{N}\Psi_\alpha(x)\Psi_\alpha(x)$ and $x_0=(x_0^i)$ be a point on the unit $(n-1)$-sphere S^{n-1}: $\delta_{ij}x^i x^j=1$[4] at which $H(x)$ attains its minimum λ^2 on S^{n-1}. It is sufficient in order to prove the theorem that $\lambda=0$.

Now, we can replace (1) by $\Psi_\alpha^*(x)\equiv\sum_{\beta=1}^{N}a_\alpha^\beta\Psi_\beta(x)=0$, where (a_α^β) is an orthogonal $N\times N$-matrix, since $H(x)=\sum_{\alpha=1}^{N}\Psi_\alpha^*(x)\Psi_\alpha^*(x)$. Accordingly, without loss of generality, we may put

$$\Psi_\alpha(x_0) = 0 \qquad (\alpha = 2, 3, \dots, N),$$

hence $\lambda=\Psi_1(x_0)$.

At the point x_0 we must have

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1) S. S. Chern and N. H. Kuiper: Some theorems on the isometric imbedding of compact Riemann manifolds in Euclidean space, Annals of Math., **56**, 422–430 (1952).

2) We shall make use of the summation convention for the repeated indices i, j, h, k.

3) S. S. Chern and N. H. Kuiper (loc. cit.), p. 427.

4) δ_{ij} is 0 if $i\neq j$, and 1 if $i=j$.

$$dH=0\,,\qquad d^2H\geqq 0$$

on S^{n-1}, which are clearly equivalent to the conditions

$$\lambda A_{1ij}x_0^i y^j=0\,, \tag{3}$$

$$\lambda A_{1ij}y^i y^j + 2\sum_{\alpha=1}^{N} A_{\alpha ih}x_0^i y^h A_{\alpha jk}x_0^j y^k \geqq 0 \tag{4}$$

for any $y\in S^{n-1}$ such that

$$\delta_{ij}x_0^i y^j=0. \tag{5}$$

Now, let us suppose that $\lambda\neq 0$. Then we get from (3), (5) the relation

$$A_{1ij}x_0^j=\lambda\delta_{ij}x_0^j. \tag{6}$$

There exists at least a point $y_0=(y_0^i)$ on S^{n-1} whose coordinates satisfy the following conditions

$$A_{\alpha ij}x_0^i y_0^j=0\,,\qquad \alpha=1,2,\dots,N \tag{7}$$

since $N<n$. By virtue of (6) and the above assumption $\lambda\neq 0$, we have $\delta_{ij}x_0^i y_0^j=0$. Hence we can substitute y_0 for y in (3), (4). Then (4) becomes $\lambda A_{1ij}y_0^i y_0^j\geqq 0$. On the other hand, we get from (2), (7) the relation $L(x_0, y_0)=\lambda A_{1ij}y_0^i y_0^j\leqq 0$. Hence we obtain the relation

$$A_{1ij}y_0^i y_0^j=0\,.$$

By the relations (7), $\varPsi_\alpha(x_0)=0$ for $\alpha>1$ and $\varPsi_1(y_0)=0$, without loss of generality, we may, in addition, put

$$\varPsi_2(y_0)=\mu\,,\quad \varPsi_\alpha(y_0)=0\,,\quad \alpha=3,4,\dots,N\,,$$

hence $H(y_0)=\mu^2$.

Now, we can represent any point on the great circle on S^{n-1} through the point x_0, y_0 by $x_0\cos\theta+y_0\sin\theta$ $(0\leqq\theta\leqq 2\pi)$, where x_0, y_0 denote also their position vectors. If we put $f(\theta)=H(x_0\cos\theta+y_0\sin\theta)$, we can easily see that

$$f(\theta)=\lambda^2\cos^4\theta+\mu^2\sin^4\theta.$$

Hence we have $f'(\theta)=4\cos\theta\sin\theta\,(-\lambda^2\cos^2\theta+\mu^2\sin^2\theta)$, $f''(\theta)=4(\cos^2\theta-\sin^2\theta)(-\lambda^2\cos^2\theta+\mu^2\sin^2\theta)+8\cos\theta\sin\theta\,(\lambda^2+\mu^2)$ where dashes denote the derivatives with respect to θ. Accordingly, we get the relations $f'(0)=0$, $f''(0)=-4\lambda^2<0$ which contradict to the assumption that $H(x)$ takes its minimum on S^{n-1} at x_0. Thus we see that $\lambda=0$. The proof is complete.