72. *The Observation Theory of the Stationary Random Process*

By Hirosi ITÔ
Department of Applied Physics, Faculty of Engineering, Osaka University
(Comm. by M. MASIMA, M.J.A., July 13, 1953)

1. Introduction

In a previous paper[1], we have discussed the influence of the fluctuation of the delay time on the spectral density of the stationary random process. The obtained result has the simple form and will be useful for the practical purpose. However, the standard deviation σ will not be determined exactly in the simultaneous observation in both time and frequency axes. If we consider the measurement in the frequency axis, it is concluded that fluctuation of the delay frequency influences on the energy density. Combining these two explicit results, we can define the quantity H which resembles, in some sense, the entropy of the discrete information theory, or the Hamiltonian of the harmonic oscillator in quantum mechanics. H has the half integer value when the expected value is considered in phase space. This expected value denotes how many elementary pulses this wave packet has. The difference between the simultaneous observation and the partial observation is discussed, and this discrepancy is seemed to be just the same as that between quantum mechanics and classical mechanics.

2. Gauss Transformation and the Frequency Autocorrelation Function

The result (2.8) in a previous paper[1] is easily obtained by the Gauss transformation. If we assume the ergodic property, we have

$$\begin{aligned}\varphi^*(\tau) &= E\{f(t+\tau+S)\overline{f(t)}\} \\ &= E_x\{E_{S=x}\{f(t+\tau+x)\overline{f(t)}\}\}\,. \end{aligned} \tag{2.1}$$

When the fluctuation of the delay time x follows the Gauss distribution where the mean value is zero and standard deviation is σ, (2.1) turns to be

$$\varphi^*(\tau) = \int_{-\infty}^{\infty} \varphi(\tau + x) \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} dx\,. \tag{2.2}$$

Fourier transform of (2.2) is

$$\Phi^*(\omega) = \Phi(\omega)\, e^{-\frac{\omega^2\sigma^2}{2}} \tag{2.3}$$

Thus we have obtained the same result as that in the previous paper[2].

The frequency autocorrelation function is defined as

$$G(\omega) = \lim_{T\to\infty}\int_{-\infty}^{\infty} \frac{A_T(u+\omega)\overline{A_T(u)}}{2T}\, du \,. \tag{2. 4}$$

This function was used by Woodward[3] for the discussion of the uncertainty relation in the measurement of the wave packet. However, he used the normalized and quadratically integrable function; but (2.4) is given more generally for the stationary random function. If we also use the Gauss transformation for the frequency autocorrelation function, we have

$$|\psi^*(t)|^2 = |\psi(t)|^2 e^{-\frac{\sigma'^2 t^2}{2}} \,. \tag{2. 5}$$

σ' is the standard deviation of the fluctuation of the delay frequencies, and $\psi(t)$ is defined by

$$\psi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(\omega)\, e^{i\omega t}\, d\omega \,, \tag{2. 6}$$

where $|\Psi(\omega)|^2 = \Phi(\omega)$. Note that $\psi(t)$ satisfies the next relation

$$\int_{-\infty}^{\infty} |\psi(t)|^2\, dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2\, dt = \text{average power.}$$

(2.5) has the following physical meaning: If we know the frequency autocorrelation function $G(\omega)$, the value of $|\psi(t)|^2$ in any time is perfectly determined by

$$|\psi(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega \,.$$

However, the fluctuation in the measurement of the frequency hinders the exact prediction in the future or in the past.

3. The Simultaneous Observation

If we put

$$x(t) = \left|\frac{\psi^*(t)}{\psi(t)}\right|^2 = e^{-\frac{\sigma'^2 t^2}{2}} \tag{3. 1}$$

$$y(\omega) = \left|\frac{\Phi^*(\omega)}{\Phi(\omega)}\right| = \left|\frac{\Psi^*(\omega)}{\Psi(\omega)}\right|^2 = e^{-\frac{\sigma^2\omega^2}{2}} \,, \tag{3. 2}$$

where $\Psi(\omega)$ is the Fourier transform of $\psi(t)$.

We have treated the case where the measurements of the time and the frequency were independent in § 2. There is an interesting case where the both measurements are tried simultaneously.

Then, the following relation will issue

$$\sigma\sigma' = 1 \tag{3. 3}$$

because of the property of the Gauss distribution. In this case, it is proved that

$$\iint_{-\infty}^{\infty} x(t)\, y(\omega)\, dt\, d\omega = 2\pi\,. \tag{3. 4}$$

$x(t)\cdot y(\omega)$ has the Gaussian wave forms in time and frequency axes respectively. This is considered as the smallest elementary pulse which we can observe in the simultaneous observation, and the relation (3.4) denotes that this elementary pulse occupies the area 2π in the phase space. Then, the quantity,

$$\begin{aligned} H &= -\log x(t)\, y(\omega) = \log \left| \frac{\psi(t)\, \Psi(\omega)}{\psi^*(t)\, \Psi^*(\omega)} \right| \\ &= \frac{(\sigma\sigma')^2}{2} \left\{ \left(\frac{t}{\sigma}\right)^2 + \left(\frac{\omega}{\sigma'}\right)^2 \right\} = \frac{1}{2} \left\{ \left(\frac{t}{\sigma}\right)^2 + \left(\frac{\omega}{\sigma'}\right)^2 \right\} \end{aligned} \tag{3. 5}$$

is defined. H may be considered as a sort of measure of the discrepancy between the true value and the observed value. If H is constant, (3.5) may be written as

$$\frac{t^2}{2C\sigma^2} + \frac{\omega^2}{2C\sigma'^2} = 1\,.$$

This is the equation of the ellipse of which area is

$$S = 2\pi\sigma\sigma' C = 2\pi C\,.$$

Consequently, C is the number of elementary pulses which are contained in this area S.

Now we define the effective time duration T and the effective bandwidth W by

$$T^2 = \frac{\int \overline{\psi(t)}\, t^2 \psi(t)\, dt}{\int |\psi(t)|^2 dt}\,; \quad W^2 = \frac{\int \overline{\Psi(\omega)}\, \omega^2 \Psi(\omega)\, d\omega}{\int |\Psi(\omega)|^2 d\omega}\,,$$

where the mean values are chosen as zero respectively. By the relation

$$\overline{H} = \frac{WT}{\sigma\sigma'} - \frac{1}{2}\left(\frac{T}{\sigma} - \frac{W}{\sigma'}\right)^2,$$

H may be considered as the number of the elementary pulses contained in this wave packet in question, if the next condition is satisfied:

$$\frac{T}{\sigma} = \frac{W}{\sigma'}\,. \tag{3. 6}$$

Of course, as we are concerned only with the area of the elementary pulse, it may be assumed that the relation (3.6) is satisfied by choosing the form of the elementary pulse similar to that of the observed wave packet. The relation (3.6) is the same as the result which was obtained by Woodward. However, the Heisenberg's uncertainty relation with respect to the elementary pulse is conserved in this case, while it breaks down in Woodward's theory.

Then we consider how to minimize the product $W.T$, or, what a wave packet has the minimum number of the elementary pulses when the average power of the time series is kept constant. This problem is reduced to a variational principle to minimize the following quantity,

$$L = \iint_{-\infty}^{\infty} \overline{\psi(t)\,\Psi(\omega)}\, H\psi(t)\,\Psi(\omega)\,dt\,d\omega - \lambda\iint_{-\infty}^{\infty} |\psi(t)\,\Psi(\omega)|^2\,dt\,d\omega\,,$$

where λ is the Lagrange's multiplier.

If the function $\psi(t)$ which satisfies $\delta L=0$ is chosen, we have

$$\lambda = \frac{\displaystyle\iint_{-\infty}^{\infty} \overline{\psi(t)\,\Psi(\omega)}\, H\psi(t)\,\Psi(\omega)\,dt\,d\omega}{\displaystyle\iint_{-\infty}^{\infty} |\psi(t)\,\Psi(\omega)|^2\,dt\,d\omega}$$

$$= \frac{\displaystyle\iint_{-\infty}^{\infty} \Psi(t)\,\frac{1}{2}\left(\frac{t}{\sigma}\right)^2 \psi(t)\,dt}{\displaystyle\int_{-\infty}^{\infty} |\psi(t)|^2\,dt} + \frac{\displaystyle\iint_{-\infty}^{\infty} \overline{\Psi(\omega)}\,\frac{1}{2}\left(\frac{\omega}{\sigma'}\right)^2 \Psi(\omega)\,d\omega}{\displaystyle\int_{-\infty}^{\infty} |\Psi(\varepsilon)|^2\,d\omega}\,.$$

Therefore, λ is the expected value of H, just as the quantum mechanical operator has its eigenvalue.

From the relations

$$\Psi(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(t)\,e^{-i\omega t}\,dt\quad \frac{d^2\Psi(\omega)}{d\omega^2}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-t^2)\,\psi(t)\,e^{-i\omega t}\,dt\,,$$

we have

$$t^2\psi(t) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d^2\Psi(\omega)}{d\omega^2}\,e^{i\omega t}\,d\omega\,.$$

Remembering these relations, it follows that

$$L=\int_{-\infty}^{\infty}|\Psi(\omega')|^2 d\omega'\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\omega-\omega'')}\,dt\iint_{-\infty}^{\infty}\overline{\Psi(\omega'')}\left(-\frac{1}{2\sigma^2}\right)\frac{d^2}{d\omega^2}\Psi(\omega)d\omega''d\omega$$

$$+\int_{-\infty}^{\infty}|\psi(t')|^2dt'\int_{-\infty}^{\infty}\overline{\Psi(\omega)}\left(\frac{\omega^2}{2\sigma'^2}\right)\Psi(\omega)d\omega-\lambda\int_{-\infty}^{\infty}|\psi(t)|^2dt\int_{-\infty}^{\infty}|\Psi(\omega)|^2d\omega$$

$$=\int_{-\infty}^{\infty}|\Psi(\omega')|^2d\omega'\int_{-\infty}^{\infty}\overline{\Psi(\omega)}\left(-\frac{1}{2\sigma^2}\frac{d^2}{d\omega^2}+\frac{\omega^2}{2\sigma'^2}-\lambda\right)\Psi(\omega)d\omega\,.$$

Under the condition

$$\int_{-\infty}^{\infty}|\psi(t)|^2dt=\int_{-\infty}^{\infty}|\Psi(\omega)|^2d\omega=\text{const.}=P\,,$$

it becomes

$$\delta L=0=P\int_{-\infty}^{\infty}\delta\overline{\Psi(\omega)}\left(-\frac{1}{2\sigma^2}\frac{d^2}{d\omega^2}+\frac{\omega^2}{2\sigma'^2}-\lambda\right)\Psi(\omega)\,d\omega\,.$$

The function $\Psi(\omega)$ which minimizes L, satisfies the following differential equation.

$$\frac{d^2\Psi(\omega)}{d\omega^2}+\left(2\sigma^2\lambda-\frac{\sigma^2}{\sigma'^2}\,\omega^2\right)\Psi(\omega)=0\,. \tag{3.7}$$

By this equation, we have

$$2\sigma^2\lambda=\frac{\sigma}{\sigma'}(2n+1)$$

or

$$\lambda = n + \frac{1}{2} \quad (n = 0,1,2, \dots). \tag{3.8}$$

The eigenfunction is given by

$$\Psi_n(\omega) = (-\sigma)^n e^{-\frac{\sigma^2\omega^2}{2}} H_n(\sigma\omega).$$

Although the eigenfunction $\Psi(\omega)$ implies the unknown parameter σ, the eigenvalue (3.8) does not contain it. The equation (3.7) is the wave equation of the harmonic oscillator in quantum mechanics, and therefore, the condition (3.6) corresponds to the law of equipartition of the energy in the theory of the harmonic oscillator when t and ω are respectively replaced by momentum p and position q of the particle; σ^2 and $(1/\sigma')^{1/2}$ by mass m and angular frequency ω_0. As H has the integral expected value except the additive constant $\frac{1}{2}$, it formally corresponds to the entropy in the discrete information theory. However, the concept of this H is based on the elementary pulse in which one can code only one information.

In a previous paper[4], we have obtained the limiting case of Wiener's prediction theory by the minimum principle of the entropy of the error function. The method which has been expounded in this paper leads us to an interesting result when being applied to the partial observation. The "partial entropy" H_ω is defined by

$$H_\omega = -\log y(\omega) = \log \Phi(\omega) - \log \Phi^*(\omega).$$

The variational principle which minimizes H_ω under the condition

$$\int \Phi(\omega)\, d\omega = P$$

requires the usual classical equation for the harmonic oscillator

$$\left(\frac{1}{\sigma'^2}\frac{\partial^2}{\partial t^2} + 2\lambda\right)\psi(t) = 0.$$

The solution of this equation gives

$$\psi_\lambda(t) = a_\lambda \,{}^{\sin}_{\cos}\,(\sqrt{2\lambda}\,\sigma' t)$$

where real a_λ satisfies

$$|a_\lambda|^2 = P.$$

Consequently, λ is permitted to take a whatever value while a_λ is determined by $\sqrt{P}$. We may understand that simultaneous observation corresponds to quantum mechanics and partial observation does to classical mechanics of the harmonic oscillator, and the previous method subjects to the partial observation and Wiener's R.M.S. method to the simultaneous one in some sense.

4. Conclusions

Althogh the formalism seems to be succeeded in, some problems will be left unsolved. The factorization of the spectral density $\Phi(\omega)$ into $|\Psi(\omega)|^2$ does not give the unique result. If we employ the method of the Wiener's prediction theory, $\Psi(\omega)$ is an analytic function in the upper half plane in the complex frequency space, and $\psi(t)$ vanishes when t is negative. Of course, as we have the same Hermite differential equation with respect to $\psi(t)$, it ranges from $-\infty$ to $+\infty$, and does not satisfy this condition. To introduce this condition in our theory, it will be necessary to take into account the Wiener-Paley's condition

$$\int_{-\infty}^{\infty} \frac{|\log|\Phi(\omega)||}{1+\omega^2} d\omega < \infty$$

in the variational principle. However, the relation (3.6) will not be conserved, because perhaps an other differential equation will appear. Then we shall be obliged to change the expression of H, even if its proper meaning would not be altered[5].

Acknowledgements

The author would like to thank Profs. K. Jôh and K. Husimi for their valuable discussions and Prof. K. Aoyagi, Dr. Z. Kiyasu, and other members of the Electrical Communication Laboratory, Nippon Telegraph Public Cooperation for their favour giving me the chance to study this problem.

References

1) H. Itô: Proc. Japan Acad., **29**, 198 (1953).
2) Dr. K. Kunisawa suggested the author the ergodic property.
3) P. M. Woodward: Phil. Mag., **42**, 883 (1951).
4) H. Itô: Proc. Japan Acad., **29**, 194 (1953).
5) Another importance to introduce this condition is to rescue the "overlap of the future" or the "breakdown of causality" which is inevitable when the elementary signal is represented by the Gaussian wave form as pointed out by D. Gabor. Jour, Inst. Elect. Eng., **93**, 429 (1946).