95. Note on the Fibering of an $(n-1)$-connected Space by Spheres

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§ 1. The main object of this note is to study an $(n-1)$-connected space, whose fibering by spheres is impossible. This is somewhat concerned with a problem of D. Montgomery and H. Samelson[1], on the fibering of a Euclidean space by compact fibres.

§ 2. Let X be an $(n-1)$-connected space $(n>2)$; namely it satisfies the conditions $\pi_i(X)=0$ $(i=0,1,\dots,n-1)$.[2] We shall also assume that X is a fibre bundle[3], whose fibre is a $(k-1)$-sphere S^{k-1} $(n>k>1)$; and let us denote whose base space as Y. Now, we shall denote following J. H. C. Whitehead[4], with ΔY the minimum dimensionality of all CW-complexes which dominate Y.[5]

If $p: X\to Y$ is the projection, we obtain an exact sequence of the homotopy groups

$$(1)\qquad \begin{array}{ccccccc} \cdots\to\pi_i(X)\to & \pi_i(X,S_0^{k-1}) & \xrightarrow{\partial} & \pi_{i-1}(S_0^{k-1}) & \to\pi_{i-1}(X)\to\cdots \\ & \downarrow p_* & & & \\ & \pi_i(Y), & & & \end{array}$$

where ∂ is the boundary operator, p_* is the isomorphism induced by p, and S_0^{k-1} is a fixed fibre oriented suitably. From the exactness of (1) and from the assumption on X, we obtain the following isomorphism onto:

$$\partial\, p_*^{-1}: \pi_i(Y)\to\pi_{i-1}(S_0^{k-1}) \quad (2\leqq i\leqq n-1).$$

Next, let E^k be an n-dimensional oriented cell, and S^{k-1} be the boundary sphere of E^k oriented coherently with E^k; let $f: S^{k-1}\to S_0^{k-1}$ be a homeomorphism of degree $+1$. Let $g: (E^k, S^{k-1})\to(S^k, s_0)$ be a mapping onto a k-dimensional oriented sphere S^k such that $g\mid \mathrm{Int}\, E^k$ is a homeomorphism of degree $+1$, and $g(S^{k-1})=s_0$, where s_0 is a fixed point on S^k. From these mappings and from (1), we obtain the following diagramm:

$$(2)\qquad \begin{array}{ccccc} \pi_i(S^k) & \xleftarrow{g_*} & \pi_i(E^k,S^{k-1}) & \xrightarrow{\partial'} & \pi_{i-1}(S^{k-1}) \\ & & & & \downarrow f_* \\ \pi_i(Y) & \xleftarrow{p_*} & \pi_i(X,S_0^{k-1}) & \xrightarrow{\partial} & \pi_{i-1}(S_0^{k-1}). \end{array}$$

Here, ∂' is the boundary operator, which is an isomorphism onto for all $i\geqq 2$; f_* and g_* are homomorphisms induced by f and g

respectively, and f_* is an isomorphism onto for all $i\geqq 2$. On the other hand, $g_* \partial'^{-1}$ is same as Freudenthal's Einhängung[6] and is an isomorphism onto for all $2\leqq i\leqq m$, where m is an integer such that

$$(3)\quad m=\begin{cases} 2k-2 & \text{if } \pi_{2k-1}(S^k) \text{ has an element whose Hopf's invariant is 1,} \\ 2k-3 & \text{otherwise.}\end{cases}$$

Next, we shall define a mapping $h:(E^k, S^{k-1})\to(X, S_0^{k-1})$ as follows: let $h\mid S^{k-1}=f$, and h shall be an arbitrary extension of f elsewhere, whose existence can be seen from $\pi_{k-1}(X)=0$. Then, h induces the following homomorphism:

$$h_* : \pi_i(E^k, S^{k-1})\to\pi_i(X, S_0^{k-1}).$$

From the construction and from (2), we can see that h_* is a homomorphism such that $\partial h_*=f_*\partial'$. So that we can write as $h_*=\partial^{-1}f_*\partial'$ when $2\leqq i\leqq n-1$. Now, we shall define a mapping $\mu: S^k\to Y$ as follows:

$$\mu(s)=\begin{cases} phg^{-1}(s) & (s\in S^k-s_0) \\ p(S_0^{k-1}) & (s=s_0).\end{cases}$$

Then, μ is easily seen to be a continuous mapping, and from the construction, it induces the following isomorphisms onto:

$$(4)\quad \begin{aligned} &\mu_*=p_*h_*g_*^{-1}=p_*\partial^{-1}f_*\partial' g_*^{-1}: \pi_i(S^k)\to\pi_i(Y)\ (2\leqq i\leqq N), \\ &\mu_*: \pi_1(S^k)\to\pi_1(Y), \\ &N=\min(n-1, m).\end{aligned}$$

In fact, we may only see that $\pi_1(Y)=0$, as we have seen the other cases. But this can be proved by the covering homotopy theorem and by the simple connectedness of X, considering a closed curve in Y to be a homotopy of a mapping from a point.

From (4), we obtain the following result using the J. H. C. Whitehead's theorem[7]:

Proposition 1. *If $\Delta Y\leqq N$, $\mu: S^k\to Y$ is a homotopy equivalence.*

In fact, because (4) is an isomorphism onto, we may only see $k\leqq N$, as $\Delta S^k=k$. From the assumption, $k\leqq n-1$ is evident. When $k=2$, as $\pi_3(S^2)$ has an element whose Hopf's invariant is 1, we see $m=2k-2=k$ from (3). Also, when $k\geqq 3$, $m\geqq 2k-3\geqq k$ follows, which completes the proof.

§ 3. The aim of this section is the following result:

Proposition 2. *The fibering of X by S^{k-1} such that $\Delta Y\leqq N$ is possible only when $2k>n$.*

In fact, let $\mu': Y\to S^k$ be a homotopy inverse of μ, namely such a mapping that satisfies the conditions $\mu\mu'\simeq 1$, $\mu'\mu\simeq 1$. Next, let M^{2k-1} be an S^{k-1}-bundle over S^k induced by μ from X.[8]

Then we obtain easily the following diagramm with the commutativity $\mu q = p\bar{\mu}$:

$$\begin{array}{ccc} M^{2k-1} & \xrightarrow{\bar{\mu}} & X \\ q\downarrow & & \downarrow p \\ S^k & \underset{\mu'}{\overset{\mu}{\rightleftarrows}} & Y \end{array},$$

where $\bar{\mu}$ is the induced map, and q is the projection for M^{2k-1}. If $2k \leqq n$, from $2k-1 \leqq n-1$ and from the assumption on X, we obtain easily $\bar{\mu} \simeq 0$. Therefore, we get

$$(5) \qquad q \simeq \mu'\mu q = \mu' p \bar{\mu} \simeq 0 .$$

So that, q is algebraically inessential, and the Hopf's invariant $H(q)$ of q can be defined to be $H(q)=0$ from (5). On the other hand, as M^{2k-1} is a sphere bundle over S^k, it must satisfy $H(q)=\pm 1$;[9] so that, such a fibering cannot exist, which completes the proof.

As S^{2k-1} is an S^{k-1}-bundle over S^k for $k=2, 4, 8$,[10] the condition $2k > n$ cannot be taken better.

As a corollary of Proposition 2, we obtain the following result:

Proposition 3. *There does not exist a fibering of an n-dimensional Euclidean space, or an n-cell (for an arbitrary n for both cases), or n-sphere* ($n \geqq 2k$) *by* $(k-1)$*-spheres such that* $\Delta Y \leqq m$; *where m is given by* (3).

References

1) D. Montgomery and H. Samelson: Fiberings with singularities, Duke Math. Journ., **13**, 51–56 (1946).

2) $\pi_0(X)=0$ means that X is arcwise connected.

3) As for definition of fibre bundles, cf. N. E. Steenrod: The topology of fibre bundles, Princeton, 1951. We shall assume the covering homotopy theorem for X, l.c. p. 54.

4) J. H. C. Whitehead: Combinatorial homotopy. I, Bull. Amer. Math. Soc., **55**, 213–245 (1949).

5) We say that P dominates Y if and only if there exist mappings $\lambda: Y \to P$, $\lambda': P \to Y$ such that $\lambda'\lambda \simeq 1$, where 1 means the identity of Y.

6) H. Freudenthal: Über die Klassen der Sphärenabbildungen. I, Comp. Math., **5**, 297–314 (1937–'38).

7) Cf. 4), Theorem 1.

8) Cf. 3), p. 47.

9) Cf. for instance the author's: On the structure of a sphere bundle, Tôhoku Math. J., 2nd Ser., **3**, 136–139 (1951). That M^{2k-1} is simply connected, and consequently that it is orientable when $k > 2$ can be proved using the covering homotopy theorem from the simple connectedness of S^{k-1} and of S^k. If it is not orientable when $k=2$, the conclusion $H(q) \equiv 1 \pmod 2$ can be proved similarly.

10) H. Hopf: Über die Abbildung von Sphären auf Sphären niedriger Dimension, Fund. Math., **25**, 427–440 (1935).