147. *On the Type of Completely Continuous Operators*

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1. Let A be an operator on a Hilbert space and let $\boldsymbol{R}(A)$ be a von Neumann algebra generated by A (i.e., the smallest von Neumann algebra containing A). Then A is said to be of type I (II, III) if $\boldsymbol{R}(A)$ is of type I (II, III). Clearly every normal operator A is of type I where $\boldsymbol{R}(A)$ is abelian. Moreover, every operator on a finite dimensional Hilbert space is of type I. Namely the classification described above has the essential meaning for non-normal operators on infinite dimensional Hilbert spaces. We shall concentrate our attention on the following question. Which non-normal operators are of type I? The answer is not much. In our recent paper [3] we have shown that an isometry is of the type I. This note is the second step in that direction.

That is, we shall prove the following theorem.

THEOREM. *A completely continuous operator on a Hilbert space is of type I.*

The class of completely continuous operators contains two important classes, the so-called Hilbert-Schmidt class and the trace class. Let A be an operator on a Hilbert space $\boldsymbol{H}$ and let $\{\varphi_i\}$ a family of complete orthonormal vectors in $\boldsymbol{H}$. Then the quantity $\sigma(A)=(\Sigma_i ||A\varphi_i||^2)^{\frac{1}{2}}$ is independent of $\{\varphi_i\}$ and the operators A for $\sigma(A)<\infty$ form the Hilbert-Schmidt class. The product of two operators in the Hilbert-Schmidt class form the trace class. As is well known, every operator in the trace class is necessarily in the Hilbert-Schmidt class and every operator in the Hilbert-Schmidt class is necessarily completely continuous. Thus we shall obtain the following corollary.

COROLLARY. *An operator in the Hilbert-Schmidt class* (*or the trace class*) *is of type I.*

By an operator we shall mean a bounded linear transformation on a Hilbert space and for the terminology of von Neumann algebras we shall always refer to [1].

2. The first step is to decompose an arbitrary operator into type I, II and III components. If a von Neumann algebra $\boldsymbol{R}(A)$ generated by A is denoted by $\boldsymbol{M}$, it is easy to see that for each $E\in\boldsymbol{M}'$, a von Neumann algebra $\boldsymbol{M}_E$ which is the restriction of $\boldsymbol{M}$ to EH is generated by the restriction A_E of A to $\boldsymbol{EH}$. Thus, keeping in mind that there exists a unique family of mutually orthogonal

central projections E_i ($i=1,2,3$) in $\boldsymbol{M}$ such that $\boldsymbol{M}=\Sigma_i \oplus M_{E_i}$ where M_{E_1} (resp. M_{E_2}, M_{E_3}) is of type I (resp. II, III), we obtain the following

LEMMA 1. *Let A be an arbitrary operator on a Hilbert space $\boldsymbol{H}$. Then there exists a unique family of mutually orthogonal central projections E_i ($i=1,2,3$) in $\boldsymbol{R}(A)$ such that*

$$A=A_{E_1}\oplus A_{E_2}\oplus A_{E_3}$$

where A_{E_1} (resp. A_{E_2}, A_{E_3}) is of type I (resp. II, III) if E_1 (resp. E_2, E_3) $\neq 0$.

This shows that certain problems dealing with arbitrary operators are reduced to the case of operators of type I (II, III).

3. Let A be an operator on a Hilbert space $\boldsymbol{H}$ and let λ a proper value of A. The null space of the operator $A-\lambda I$ is called the proper subspace of A corresponding to λ and is denoted by $\mathfrak{N}_A(\lambda)$. That is to say, $\mathfrak{N}_A(\lambda)=\{\varphi\in\boldsymbol{H};\ A\varphi=\lambda\varphi\}$. Then it is well known that a proper subspace of a completely continuous operator corresponding to a non-zero proper value is finite dimensional. Actually, this fact gives the key to the proof of the theorem. The following lemma can be viewed as the natural step toward the proof.

LEMMA 2. *Let $\boldsymbol{M}$ be a von Neumann algebra generated by an operator A on a Hilbert space $\boldsymbol{H}$ and let E a projection on the proper subspace $\mathfrak{N}_A(\lambda)$ of A. Then E belongs to $\boldsymbol{M}$.*

PROOF. It is sufficient to show that E commutes with all operators belonging to $\boldsymbol{M}'$. Let A' be an arbitrary operator in M'. Then, for all $\varphi\in\mathfrak{N}_A(\lambda)$, the equality $A(A'\varphi)=A'A\varphi=A'\lambda\varphi=\lambda A'\varphi$ yields $A'\varphi\in\mathfrak{N}_A(\lambda)$. Similarly, $A'^*\varphi\in\mathfrak{N}_A(\lambda)$ for all $\varphi\in\mathfrak{N}_A(\lambda)$, thus $\mathfrak{N}_A(\lambda)$ reduces A'. This means that E commutes with A'.

Before beginning to prove the theorem, we need to mention the special property of a self-adjoint completely continuous operator A on a Hilbert space $\boldsymbol{H}$. It is well known that there exists an orthonormal basis in $\boldsymbol{H}$ whose elements are proper vectors of A (cf. [2; Theorem 6]). Therefore, let $\{\mu_i\}$ be a family of all distinct proper values of A (which is necessarily a finite or a denumerably infinite sequence) and let P a projection on the proper subspace $\mathfrak{N}_A(\mu_i)$. Then $\Sigma_i P_i=I$ and $A=\Sigma_i\mu_i P_i$. We are now in a position to prove the theorem of this paper.

PROOF OF THEOREM. Let A be a completely continuous operator on a Hilbert space $\boldsymbol{H}$. Then $[A]=(A^*A)^{\frac{1}{2}}$ is also completely continuous. Therefore the self-adjoint operator $[A]$ admits the following representation as mentioned above; $[A]=\Sigma_i\mu_i P_i$, where P_i is the projection on the proper subspace $\mathfrak{N}_{[A]}(\mu_i)$ and $\Sigma_i P_i=I$. It follows from Lemma 2 that each P_i belongs $\boldsymbol{R}([A])\subset\boldsymbol{R}(A)$. Moreover, if P_i is a projection corresponding to a non-zero proper value μ_i, P_iH is finite dimensional since $[A]$ is completely continuous.

Now, by Lemma 1, A is decomposed in the form $A=A_E\oplus A_F$ where E and F are mutually orthogonal central projections of $\boldsymbol{R}(A)$ such that $E+F=I$ and A_E is of type I if $E\neq 0$ and A_F generates a continuous von Neumann algebra if $F\neq 0$.

To prove our assertion we must show that $F=0$. Let us suppose $F\neq 0$. Since F is a central projection in $\boldsymbol{R}(A)$, $\{FP_i\}$ is a family of mutually orthogonal projection in $\boldsymbol{R}(A)$ and $F=\Sigma_i FP_i$. If there exists a projection P_k corresponding to a non-zero proper value μ_k such that $FP_k\neq 0$, $\boldsymbol{R}(A)FP_k$ is of type I since $FP_k\boldsymbol{H}$ is finite dimensional. This contradicts to the fact that $\boldsymbol{R}(A)_F$ is a continuous von Neumann algebra. Thus $FP_k=0$ if $\mu_k\neq 0$. Consequently, if there exists a projection P_0 corresponding to the proper value 0, $F=FP_0$, and otherwise $F=0$. The equality $F=FP_0$ implies that FH is contained in the null space of $[A]$ (that is, A). Hence $A_F=0$ and $\boldsymbol{R}(A_F)$ is of type I. This contradiction shows that $F=0$. Now we can conclude that $\boldsymbol{R}(A)$ is of type I, that is to say, A is of type I.

References

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[3] N. Suzuki: Isometries on Hilbert spaces. Proc. Japan Acad., **39**, 435–438 (1963).