217. *Notes on (E. R.)-Integrals*

By Sumiyuki KOIZUMI

Department of Applied Mathematics, Osaka University

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§1. **Introduction.** The (E. R.)-integral has its own history of more than ten years. It began when Prof. K. Kunugi [3][1] introduced the notion of *l'espace rangés* as a generalization of metric space and he defined an integral as a realization of complete ranked space. Since then, researches of (E. R.)-integral and its application have been continued by the founder and followers. Contributions to Fourier analysis are main purpose of this paper.

§2. **The ranked space.** Let us locate the notion of ranked space briefly which we use in the following sections. Detailed theory is found out in works of K. Kunugi [3–5]. Let us start with the Hausdorff space R. Then if for every real positive number ε, there correspond a family of neighbourhood $\mathfrak{A}_\varepsilon$, we shall define it as a *ranked space*. Let us consider a monotone decreasing sequence of neighbourhoods

$$v_1(p_1)\supseteq v_2(p_2)\supseteq\cdots\supseteq v_n(p_n)\supseteq\cdots \tag{2.1}$$

which satisfy the following properties: each neighbourhood $v_n(p_n)$ belongs to the family $\mathfrak{A}_{\varepsilon_n}$ and sequence of positive numbers $\{\varepsilon_n\}$ tend monotonously to zero. We shall define it as a *Cauchy sequence*. The ranked space will be called *complete* if for every Cauchy sequence $\bigcap v_n(p_n)$ has no empty intersection property: that is $\bigcap v_n(p_n)\neq\phi$. Here we only state the ranked space of depth $\aleph_0$. Prof. K. Kunugi proved the following big theorems in general case.

Theorem A. *For the metric space is to be complete, it is necessary and sufficient that it is complete if we consider it as the ranked space.*

Theorem B. *On the complete ranked space, every non empty open set can not be first category.*

Thus we can understand the ranked space as a natural extension of metric space. For fix our ideas, let us consider on the real line and the family of simple functions which are linear combinations of characteristic functions on finite intervals. Then if we complete this family considering as a ranked space, we obtain the new family of functions containing the Lebesgue integrable functions as a true sub-class.

§3. **(E.R.)-integrals and its properties.** We shall describe fundamental results which we use in the following sections. Let us start with a set M, the vector space of real valued measurable functions to be defined on a finite interval $I=(a, b)$. For any positive number ε and any measurable sub-set A of I, we shall denote $V(\varepsilon, A, f)$, the ε-neighbourhood of f over A, that is the set of functions $g \in M$ such that

$$\int_A |f(x)-g(x)|\,dx \leqq \varepsilon. \tag{3.1}$$

Let us denote by CA the complement of set A and by an absolute value the Lebesgue measure. If it is satisfied the following additional condition:

$$(F_1) \qquad |CA| \leqq \varepsilon,$$

then we shall define $\mathfrak{A}_\varepsilon$ as the family of neighbourhoods $V(\varepsilon, A, f)$ with any $A \subset I$ and any $f \in M$.

Let us consider sequence of neighbourhoods $\{V(\varepsilon_n, A_n, f_n)\}$ where each $V(\varepsilon_n, A_n, f_n)$ is picked up from the family $\mathfrak{A}_{\varepsilon_n}$ respectively. Moreover let us suppose that it is monotone decreasing:

$$V(\varepsilon_1, A_1, f_1) \supseteq V(\varepsilon_2, A_2, f_2) \supseteq \cdots \supseteq V(\varepsilon_n, A_n, f_n) \supseteq \cdots \tag{3.2}$$

and the $\{\varepsilon_n\}$ tends monotonously to zero. We shall define these as Cauchy sequence.

Theorem C. *If the sequence of neighbourhood $\{V(\varepsilon_n, A_n, f_n)\}$ is that of Cauchy, then it determines one and only one function $f \in M$ except a set of measure zero. Furthermore we have.*

$$\int_{A_n} |f(x)-f_n(x)|\,dx \leqq \varepsilon_n, \qquad (n=1, 2, \cdots) \tag{3.2}$$

and f_n tend to f a.e.

Let us introduce two more conditions:

(F_2) for any measurable set B such that $|B| \leqq |CA_n|$, it is satisfied

$$\int_B |f_n|\,dx \leqq \varepsilon_n .$$

(P^*) there exist a constant $k>1$ such as $|CA_n| \leqq k|CA_{n+1}|$ for every n.

Theorem D. *Let us considere two Cauchy sequence $\{V(\varepsilon_n, A_n, f_n)\}$ and $\{V(\eta_n, B_n, g_n)\}$ which tend to the same function f. Let us assume that conditions (P^*) and (F_2) are satisfied. Then we have*

$$\overline{\lim_{n\to\infty}} \int_I f_n(x)\,dx = \overline{\lim_{n\to\infty}} \int_I g_n(x)dx \tag{3.3}$$

respectively.

By this theorem, if the limit of (3.3) exist and finite, it is natural to define it as the value of integral of f. Then f is called (E. R)-integrable and write

$$\begin{aligned}\text{(E. R.)} \int_I f(x)\,dx &= \lim_{n\to\infty} \int_I f_n(x)dx \\ &= \lim_{n\to\infty} \int_{A_n} f(x)dx\end{aligned} \tag{3.4}$$

where the last equality is followed from the (3.2). Mr. H. Okano (7) denotes by K the class of function f which is determined by the Cauchy sequence and by K^* the class of (E. R.)-integrable functions.

Theorem E. *The class K^* is a vector space.*

Prof. I. Amemiya intended the measure-theoretic approach to this singular integral and proposed a simplified definition. Let us denote $f^{(n)}$ the truncation of f, that is $f^{(n)}(x)=f(x)$, if $|f|\leqq n$; $=n$ sign f, if $|f|>n$. Then if it satisfies that

(i) $$|\{x \mid |f(x)|>n\}|=o(n), \quad \text{as } n\to\infty$$

(ii) $$\lim_{n\to\infty}\int_I f^{(n)}(x)dx \text{ exist and finite,}$$

it is called that f is integrable. With Mr. Ando [1, 2], they proved that this definition is equivalent to that of (E. R.)-integral. Then unfortunately they were never aware of the existence of the (A)-integrals [11] of Russian mathematicians. But they introduced the following quasi-norm

(3.5) $$\|f\|_*=\sup_{\gamma>0}\left|\int_I f^{(\gamma)}(x)dx\right|+\sup_{\gamma>0}\gamma|\{x \mid f(x)\geqq\gamma\}|$$

and characterized the (E. R.)-integral.

Theorem F. *The class K^* is complete with respect to the quasi-norm and contains the class L as a dense sub-set.*

We should notice that (A)-integrals are just the same as (Q)-integrals of E. C. Titchmarsh [8] and integrals of type B introduced by A. Denjoy [9, 10], are associated with it.

§4. Its generalization. (E. R.)-integrals of type ν. The function $1/x|\log|x||$ is (E. R.)-integrable on the interval $(-1, 1)$, but not the function $1/x$. Prof. K. Kunugi proposed the generalization of the integral. It was solved by Mr. H. Okano [7]. Let us introduce a finite measure ν and suppose following properties:

(i) measure ν is defined on every Lebesgue measurable sub-set of the interval I.

(ii) $\nu(A)=0$ if and only if $|A|=0$.

Let us also introduce conditions:

$F_1(\nu)$ $\nu(CA_n)\leqq\varepsilon_n$

$F_2(\nu)$ for any measurable set B such as $\nu(B)\leqq\nu(CA_n)$ it is satisfied $\int_B|f_n|dx\leqq\varepsilon_n$.

$P^*(\nu)$ there exists a constant $k>1$ such as $\nu(CA_n)\leqq k\nu(CA_{n+1})$ $(n=1, 2, \cdots)$.

We shall define the Cauchy sequence $\{V(\varepsilon_n, A_n, f_n)\}$ as precedings and (E. R.)-integral:

(4.1) $$\text{(E. R.)}\int_I f(x)dx=\lim_{n\to\infty}\int_I f_n(x)dx.$$

We should notice the fact that the limit does not relate with the ν-measure. It is also remarkable that (E. R.)-integrals are invariant under measure preserve transformation.

§5. (E. R.)-integrals of order $p(p>1)$. On some problems in the Fourier analysis we need to consider an f which is approximated by the sequence $\{f_n\}$ which belong to the class $L_p(p>1)$ instead of L_1 as precedings. Let us denote the interval $(-\pi, \pi)$ by I. For any positive number and any measurable sub-set A of I, we define a ε-neighbourhood of f over A, $V_p(\varepsilon, A, f)$ which consist of functions g such that

$$(5.1)\qquad |2\pi|^{\frac{1}{q}}\left(\int_A |f(x)-g(x)|^p\,dx\right)^{\frac{1}{p}} \leqq \varepsilon,$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$. Then we have (c.f. H. Okano [6]).

Lemma 1. *For the sequence of neighvourhoods $\{V_p(\varepsilon_n, A_n, f_n)\}$ with $\{\varepsilon_n\}\downarrow 0$, to be monotone decreasing, it is necessary and sufficient that the following conditions are satisfied*

$$(5.2)\qquad \nu(A_n-A_{n+1})=0,$$

$$(5.2)\qquad |2\pi|^{\frac{1}{q}}\left(\int_{A_n} |f_n(x)-f_{n+1}(x)|^p dx\right)^{\frac{1}{p}} \leqq \varepsilon_n-\varepsilon_{n+1}$$

where $n=1, 2, \cdots$.

Proof. Necessity. We suppose to the contrary that $|A_n-A_{n+1}|=\delta>0$, and define $g(x)=f_{n+1}(x)$, if $x\in A_{n+1}$; $=f_n(x)+\eta$, if $x\in A_n-A_{n+1}$; $=0$, elsewhere. Then we have $|2\pi|^{\frac{1}{q}}\left(\int_{A_{n+1}} |g-f_{n+1}|^p dx\right)^{\frac{1}{p}}=0$ and g belongs to $V_p(\varepsilon_{n+1}, A_{n+1}, f_{n+1})$. On the other hand we have

$$|2\pi|^{\frac{1}{q}}\left(\int_{A_n} |g-f_n|^p dx\right)^{\frac{1}{p}} \geqq |2\pi|^{\frac{1}{q}}\left(\int_{A_n-A_{n+1}} |g-f_n|^p dx\right)^{\frac{1}{p}} = |2\pi|^{\frac{1}{q}}\eta|A_n-A_{n+1}|$$

and if we take $\eta=2\varepsilon_n/|2\pi|^{\frac{1}{q}}|A_n-A_{n+1}|$ then we obtain

$$|2\pi|^{\frac{1}{q}}\left(\int_{A_n} |g-f_n|^p dx\right)^{\frac{1}{p}} \geqq 2\varepsilon_n$$

and it reads to the contradition $g\notin V_p(\varepsilon_n, A_n, f_n)\supseteq V_p(\varepsilon_{n+1}, A_{n+1}, f_{n+1})$. Next as for the (5.2), we define $g(x)=f_{n+1}(x)$, if $x\in CA_n$, and if $x\in A_n$ by the geometrical consideration we can find out $g(x)$ such that f_n-g_{n+1} and $f_{n+1}-g$ have the same signe and proportional, among these g we can select a g such that $|2\pi|^{\frac{1}{q}}\left(\int_{A_{n+1}} |f_{n+1}-g|^p dx\right)^{\frac{1}{p}}=e_{n+1}$. Then it reads that $|f_n-g|\sim|f_n-f_{n+1}|$ and $|f_n-g|\sim|f_{n+1}-g|$, where notation $\sim$ means to be propositional to each other. Therefore we have

$$\begin{aligned}
&\int_{A_n} |f_n-g|^p dx \\
&=\int_{A_n} |f_n-g|^{p-1}\{|f_n-f_{n+1}|+|f_{n+1}-g|\}dx \\
&=\left(\int_{A_n} |f_n-g|^p\,dx\right)^{\frac{1}{q}}\left\{\left(\int_{A_n} |f_n-f_{n+1}|^p dx\right)^{\frac{1}{p}}+\left(\int_{A_n} |f_{n+1}-g|^p dx\right)^{\frac{1}{p}}\right\}
\end{aligned}$$

and

$$\left(\int_{A_n}|f_n-g|^p dx\right)^{\frac{1}{p}}=\left(\int_{A_n}|f_n-f_{n+1}|^p dx\right)^{\frac{1}{p}}+\left(\int_{A_{n+1}}|f_{n+1}-g|^p dx\right)^{\frac{1}{p}}$$

and by hypothesis $g\in V_p(\varepsilon_{n+1}, A_{n+1}, f_{n+1})\subset V_p(\varepsilon_n, A_n, f_n)$ we obtain the condition (5.2). The sufficiency of conditions are clear.

Lemma 2. *Under the same hypothesis as Lemma* 1, *there exist a musurable function f such that f_n tends to f a.e. on each set A_n and it is satisfied*

$$|2\pi|^{\frac{1}{q}}\left(\int_{A_n}|f-f_n|^p dx\right)^{\frac{1}{p}}\leqq\varepsilon_n. \tag{5.3}$$

Proof. It is immediate by the condition (5.1) and (5.2).

Lemma 3. *Under the same hypothesis as Lemma* 1, *we have for the sequence of neighbourhoods $\{V(\varepsilon_n, A_n, f_n)\}$ which are defined in the preceding section,*

$$V(\varepsilon_1, A_1, f_1)\supseteq V(\varepsilon_2, A_2, f_2)\supseteq\cdots\supseteq V(\varepsilon_n, A_n, f_n)\supseteq\cdots \tag{5.4}$$

$$V(\varepsilon_n, A_n, f_n)\supseteq V_p(\varepsilon_n, A_n, f_n). \tag{5.5}$$

Proof. The assumption $g\in V(\varepsilon_{n+1}, A_{n+1}, f_{n+1})$ reads that

$$\begin{aligned}\int_{A_n}|f_n-g|dx &\leqq\int_{A_n}|f_n-f_{n+1}|dx+\int_{A_n}|f_{n+1}-g|dx\\ &\leqq|2\pi|^{\frac{1}{q}}\left(\int_{A_n}|f_n-f_{n+1}|^p dx\right)^{\frac{1}{p}}+\int_{A_{n+1}}|f_{n+1}-g|dx\\ &\leqq(\varepsilon_n-\varepsilon_{n+1})+\varepsilon_{n+1}=\varepsilon_n,\end{aligned}$$

and therefore we obtain $g\in V(\varepsilon_n, A_n, f_n)$. The remaining part is obvious.

The sequence of neighbourhood $\{V_p(\varepsilon_n, A_n, f_n)\}$ of the Lemma 1 is called a Cauchy sequence which tends to f. Combining Theorem D and Lemmas 2 and 3 we obtain

Theorem 1. *Let us consider two Cauchy sequences $\{V_p(\varepsilon_n, A_n, f_n)\}$ and $\{V_p(\eta_n, B_n, g_n)\}$ which tend to the same f and both satisfy conditions $F_2(\nu)$ and $P^*(\nu)$. Then we have*

$$\overline{\lim_{n\to\infty}}\int_I f_n(x)dx=\overline{\lim_{n\to\infty}}\int_I g_n(x)dx \tag{5.6}$$

respectively.

If we denote as before the set of functions to be defined by the Cauchy sequence by $K_p(\nu)$ and the set of functions to be (E. R.)-integrable by $K_p^*(\nu)$ respectively.

Theorem 2. *The class $K_p^*(\nu)$ is a vector space.*

We have also the following relations:

$$K_p(\nu)\subset K(\nu)\quad\text{and}\quad K_p^*(\nu)\subset K^*(\nu). \tag{5.7}$$

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