193. *Unitary Representations of SL(n, C)*

By Masao TSUCHIKAWA
Mie University

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1968)

§1. Let G be the group $SL(n, C)$ and $\chi=\{\lambda_1, \mu_1; \cdots; \lambda_{n-1}, \mu_{n-1}\}$ (λ_i and μ_i are complex numbers and $\lambda_i-\mu_i$ are integers) be a character of the diagonal subgroup $D: \chi(\delta)=(\delta_2\delta_3\cdots\delta_n)^{(\lambda_1,\mu_1)}(\delta_3\cdots\delta_n)^{(\lambda_2,\mu_2)}\cdots \cdots\delta_n{}^{(\lambda_{n-1},\mu_{n-1})}$ (where $z^{(\lambda,\mu)}=z^\lambda\bar{z}^\mu$), and W be the Weyl group G whose elements can be identified with $s=\begin{pmatrix}1\cdots n\\ j_1\cdots j_n\end{pmatrix}$ of the permutation group $\mathfrak{S}_n$ generated by $s_i=(i\ i+1)$. We divide the set of characters into two classes: regular and singular. A regular character χ is such that any one of pairs (λ_i', μ_i') contained in $\chi'=\chi^s$ for all $s\in W$ is not a pair of integers of the same signature, and a singular character χ is otherwise. Furthermore we call the singular character χ to be of type (D), if some pair in χ is $(-1, -1)$ and any other pair (λ_i', μ_i') in $\chi'=\chi^s$ for s, such that s leaves the totality of pairs $(-1, -1)$ in χ stable, is not a pair of integers of the same signature.

In this paper we shall discuss the unitarity of the elementary representation $R(\chi)=\{T^\chi, \mathcal{D}_\chi\}$ of G for a regular character χ and a singular character χ of type (D). We shall remark that the unitary representation of the degenerate supplementary series recently described by E. M. Stein [1] is one of the representations $R(\chi)$ of the latter case.

§2. Generalizing the method in [2] and [3], we consider the invariant bilinear form between two representations $R(\chi)$ and $R(\chi')$. In this point of view the integral kernel of the invariant bilinear form is analytically continued rather than the representation itself.

Let $\langle\varphi, \psi\rangle=B(\varphi, \bar{\psi})$ for φ and $\psi\in\mathcal{D}_\chi$, where $B(\cdot, \cdot)$ is an invariant bilinear form on $\mathcal{D}_\chi\times\mathcal{D}_{\bar{\chi}}$, and then $\langle\cdot, \cdot\rangle$ is an Hermitian form on $\mathcal{D}_\chi$. If $\langle\cdot, \cdot\rangle$ exists and is positive definite, the representation $R(\chi)$ is unitary with respect to this scalar product. In order that the invariant Hermitian form on $\mathcal{D}_\chi$ exists, it is necessary and sufficient that χ satisfied the condition that $\chi\bar{\chi}^s=1$ for some $s\in W$.

§3. Let χ be a regular character satisfying $\chi\bar{\chi}^s=1$ for some $s\in W$, then we have a non-degenerate Hermitian form on $\mathcal{D}_\chi$. Now if we set $\chi'=\chi^{s_i}$, then the representations $R(\chi)$ and $R(\chi')$ are equivalent by means of the intertwinning operator A_i:

$$A_i\varphi(z)=\gamma(\lambda_i, \mu_i)\int z'^{(\lambda_i-1,\mu_i-1)}_{i+1\,i}\varphi(z'_{i+1\,i}z)dz'_{i+1\,i}.$$

And χ' satisfies the condition $\chi'\bar{\chi}'^{s_i s s_i^{-1}}=1$. Namely if s and s' are conjugate elements of W, and if χ satisfies $\chi\bar{\chi}^s=1$, then we have χ' satisfying $\chi'\bar{\chi}'^{s'}=1$ such that $R(\chi')$ is equivalent to $R(\chi)$. Moreover if $\langle\cdot,\cdot\rangle$ on $\mathcal{D}_\chi$ is positive definite, i.e., if $R(\chi)$ is unitary, then $R(\chi')$ is also unitary and unitarily equivalent to $R(\chi)$ with respect to the corresponding scalar products. This shows that it is enough to consider χ satisfying the condition $\chi\bar{\chi}^s=1$ for the representative elements s of the conjugate classes of W for enumerating unitary representations $R(\chi)$ of regular characters. But there exists χ satisfying $\chi\bar{\chi}^s=1$ only for $s=e$, s_1, s_1s_3, $\cdots$, and $s_1s_3\cdots s_{2k-1}$ ($k\leq[n/2]$).

(i) When $\chi\bar{\chi}=1$, that is, $\lambda_i=(n_i+\sqrt{-1}\,\rho_i)/2$, $\mu_i=(-n_i+\sqrt{-1}\,\rho_i)/2$, where n_i are integers and ρ_i are real, then the representation $R(\chi)$ is unitary with respect to the scalar product

$$\langle\varphi,\psi\rangle=\int\varphi(z)\overline{\psi(z)}dz,$$

which is known as that of the non-degenerate principal series.

(ii) Let $\chi\bar{\chi}^{s_1s_3\cdots s_{2k-1}}=1$, that is, $\lambda_{2i-1}=\mu_{2i-1}=\sigma_{2i-1}$, $\lambda_{2i}=-\sigma_{2i-1}/2+(n_{2i}+\sqrt{-1}\,\rho_{2i})/2$, $\mu_{2i}=-\sigma_{2i-1}/2+(-n_{2i}+\sqrt{-1}\,\rho_{2i})/2$ $(i\leq k)$, and $\lambda_i=(n_i+\sqrt{-1}\,\rho_i)/2$, $\mu_i=(-n_i+\sqrt{-1}\,\rho_i)/2$ $(2k<i<n)$. Then the representation $R(\chi)$ is unitary for $-1<\sigma_{2i-1}<1$, $\sigma_{2i-1}\neq 0$ with respect to the scalar product

$$\langle\varphi,\psi\rangle=\int|z'_{21}|^{-2\sigma_1-2}|z'_{43}|^{-2\sigma_3-2}\cdots|z'_{2k\,2k-1}|^{-2\sigma_{2k-1}-2}\varphi(z'z)\overline{\psi(z)}dz'dz,$$

where $z'=z'_{21}z'_{43}\cdots z'_{2k\,2k-1}$, which is known as that of the non-degenerate supplementary series.

§4. In this section we give a remark about the degenerate representation contained in the representation $R(\chi)$ with a singular character χ of type (D).

We can canonically express χ of type (D) in the following form up to the equivalency:

$$\chi=\{\overbrace{-1,-1;\cdots;-1,-1}^{p_1-1};\lambda_{p_1},\mu_{p_1};\overbrace{-1,-1;\cdots}^{p_2-1};\lambda_{p_1+p_2},\mu_{p_1+p_2};\cdots$$
$$\overbrace{\cdots;-1,-1}^{p_k-1};\lambda_{p_1+p_2+\cdots+p_k},\mu_{p_1+p_2+\cdots+p_k};\cdots;\lambda_{n-1},\mu_{n-1}\},$$

where $p_1\geq p_2\geq\cdots\geq p_k$, and $p_1+p_2+\cdots+p_k+1+1\cdots+1=n$ is an integral partition of n.

Let us decompose a matrix $z\in Z$ into blocks z_{ij} according to the above partition of n. Every diagonal block of z is again a lower triangular matrix and we call the matrix z whose blocks vanish except for the diagonal ones the triangular diagonal matrix in the partition of n.

In the representation $R(\chi)$ we can obtain the degenerate representation as the factor representation by the subrepresentation $\{T^\chi,\mathcal{F}_\chi\}$, whose space $\mathcal{F}_\chi$ is the linear space of all functions $\varphi \in \mathcal{D}_\chi$ such that $\int \varphi(z'z)dz'=0$, where the integration is taken over the set of all triangular diagonal matrices in the above partition. This factor representation is realized by $\int \varphi(z'z)dz' \mapsto \int T_g^\chi \varphi(z'z)dz'$ for $\varphi \in \mathcal{D}_\chi$ and we simply call it the factor representation $R(\chi)$.

§5. To avoid the complexity, we divide the parameters of singular character χ of type (D) in §4 into the following types, each of which we denote again by χ.

$$\text{(a)}\quad \chi=\{\overbrace{-1, -1; \cdots}^{p-1}; \lambda_p, \mu_p; \cdots; \lambda_{n-1}, \mu_{n-1}\},$$

$$\text{(b)}\quad \chi=\{\overbrace{-1, -1; \cdots}^{p-1}; \lambda_p, \mu_p; \overbrace{-1, -1; \cdots}^{q-1}\} \qquad (p>q,\ p+q=m),$$

$$\text{(c)}\quad \chi=\{\overbrace{-1, -1; \cdots}^{p-1}; \lambda_p, \mu_p; \overbrace{-1, -1; \cdots}^{p-1}; \lambda_{2p}, \mu_{2p}; \cdots \overbrace{\cdots}^{p-1}; \lambda_{(l-1)p}, \mu_{(l-1)p}; \overbrace{-1, -1; \cdots}^{p-1}\} \qquad (lp=m).$$

Thus we reduce to the representation $R(\chi)$ of $SL(m, C)$ $(n \geq m)$ for the characters χ of the above three cases.

(a) In order that $\chi\bar{\chi}^s=1$ is satisfied for χ of (a), s should be of the form ts' where $t=\begin{pmatrix} 1 & 2 & \cdots p \\ p & p-1 \cdots & 1 \end{pmatrix}$ and s' is an element expressed by a product of s_i $(i>p)$. For the analogous reason as in §3, it is enough to consider χ satisfying $\chi\bar{\chi}^s=1$ only for $s=t, ts_{p+1}, ts_{p+1}s_{p+3}, \cdots$, and $ts_{p+1}\cdots s_{p+2k-1}$ $(p+2k<m)$.

(i) When $\chi\bar{\chi}^t=1$, that is, $\lambda_p=(p-1)/2+(n_p+\sqrt{-1}\,\rho_p)/2$, $\mu_p=(p-1)/2+(-n_p+\sqrt{-1}\,\rho_p)/2$ and $\lambda_i=(n_i+\sqrt{-1}\,\rho_i)/2$, $\mu_i=(-n_i+\sqrt{-1}\,\rho_i)/2$ $(p<i)$, then the factor representation $R(\chi)$ is unitary with respect to the scalar product

$$\langle \varphi, \psi \rangle = \int \varphi(z'z)\overline{\psi(z)}dz'dz,$$

where the integration by z' is taken over the set of all triangular diagonal matrices in the partition $m=p+1+\cdots+1$.

(ii) Let $\chi\bar{\chi}^{ts_{p+1}\cdots s_{p+2k-1}}=1$, that is, $\lambda_p=(p-1)/2+(n_p+\sqrt{-1}\,\rho_p)/2$, $\mu_p=(p-1)/2+(-n_p+\sqrt{-1}\,\rho_p)/2$ and $\lambda_{p+2i-1}=\mu_{p+2i-1}=\sigma_{p+2i-1}$, $\lambda_{p+2i}=-\sigma_{p+2i-1}/2+(n_{p+2i}+\sqrt{-1}\,\rho_{p+2i})/2$, $\mu_{p+2i}=-\sigma_{p+2i-1}/2+(-n_{p+2i}+\sqrt{-1}\,\rho_{p+2i})/2$ $(i\leq k)$ and $\lambda_{p+i}=(n_{p+i}+\sqrt{-1}\,\rho_{p+i})/2$, $\mu_{p+i}=(-n_{p+i}+\sqrt{-1}\,\rho_{p+i})/2$ $(p+2k<i<m)$. Then the factor representation $R(\chi)$ is unitary for $-1<\sigma_{p+2i-1}<1$, $\sigma_{p+2i-1}\neq 0$ with respect to the scalar product

$$\langle \varphi, \psi \rangle = \int |z'_{p+2\ p+1}|^{-2\sigma_{p+1}-2} \cdots |z'_{p+2k\ p+2k-1}|^{-2\sigma_{p+2k-1}-2}\varphi(z'z)\overline{\psi(z)}dz'dz,$$

where the integration by z' is taken over the set of all product of the triangular diagonal matrices in the partition of (i) and the matrices of the form $z'_{p+2\ p+1}z'_{p+4\ p+3}\cdots z'_{p+2k\ p+2k-1}$.

(b) In this case, there exist χ satisfying $\chi\bar{\chi}^s=1$ only for $s=\begin{pmatrix}1\cdots p\\ p\cdots 1\end{pmatrix}\begin{pmatrix}p+1\cdots & m\\ m & \cdots p+1\end{pmatrix}$. And then $\lambda_p=(m-2)/2+(n_p+\sqrt{-1}\rho_p)/2$, $\mu_p=(m-2)/2+(-n_p+\sqrt{-1}\rho_p)/2$ and the factor representation $R(\chi)$ is unitary with respect to the scalar product

$$\langle\varphi,\psi\rangle=\int\varphi(z'z)\overline{\psi(z)}dz'dz,$$

where the integration by z' is taken over the set of all triangular diagonal matrices in the partition $m=p+q$.

(c) In order that $\chi\bar{\chi}^s=1$ is satisfied for χ of (c), s should be of the form ts' where t is the product of all $t_i=\begin{pmatrix}(i-1)p+1\cdots & ip\\ ip & \cdots(i-1)p+1\end{pmatrix}$ $(1\leq i\leq l)$ and s' is an element expressed by a product of $u_i=\begin{pmatrix}(i-1)p+1\cdots & ip & ip+1 & \cdots(i+1)p\\ ip+1 & \cdots(i+1)p & (i-1)p+1\cdots & ip\end{pmatrix}$, and analogously as in §3 it is enough to consider χ satisfying $\chi\bar{\chi}^s=1$ only for $s=t$, tu_1, tu_1u_3, $\cdots$, and $tu_1\cdots u_{2k-1}$ $(k\leq[l/2])$.

(i) When $\chi\bar{\chi}^t=1$, that is, $\lambda_{ip}=(p-1)+(n_{ip}+\sqrt{-1}\rho_{ip})/2$, $\mu_{ip}=(p-1)+(-n_{ip}+\sqrt{-1}\rho_{ip})/2$ $(i<l)$, then the factor representation $R(\chi)$ is unitary with respect to the scalar product

$$\langle\varphi,\psi\rangle=\int\varphi(z'z)\overline{\psi(z)}dz'dz,$$

where the integration by z' is taken over the set of all triangular diagonal matrices in the partition $m=p+p+\cdots+p$.

(ii) Let $\chi\bar{\chi}^{tu_1u_2\cdots u_{2k-1}}=1$, that is, $\lambda_{(2i-1)p}=\mu_{(2i-1)p}=(p-1)+\sigma_{(2i-1)p}$ and $\lambda_{2ip}=(p-1)-\sigma_{(2i-1)p}/2+(n_{2ip}+\sqrt{-1}\rho_{2ip})/2$, $\mu_{2ip}=(p-1)-\sigma_{(2i-1)p}/2+(-n_{2ip}+\sqrt{-1}\rho_{2ip})/2$ $(i\leq k)$ and $\lambda_{ip}=(p-1)+(n_{ip}+\sqrt{-1}\rho_{ip})/2$, $\mu_{ip}=(p-1)+(-n_{ip}+\sqrt{-1}\rho_{ip})/2$ $(2k<i<l)$. Then the factor representation $R(\chi)$ is unitary for $-1<\sigma_{(2i-1)p}<1$, $\sigma_{(2i-1)p}\neq 0$ with respect to the scalar product

$$\langle\varphi,\psi\rangle=\int|det(z'_{21})|^{-2p-2\sigma_p}\cdots|det(z'_{2k\ 2k-1})|^{-2p-2\sigma_{(2k-1)p}}\varphi(z'z)\overline{\psi(z)}dz'dz,$$

where the integration by z' is taken over the set of all matrices whose blocks in the partition $m=p+p+\cdots+p$ vanish except for the triangular diagonal part and the blocks z'_{21}, $\cdots$, and $z'_{2k\ 2k-1}$.

The unitary representations $R(\chi)$ of (a i), (b), and (c i) are known as those of the degenerate principal series. The unitary representations $R(\chi)$ of (a ii) and (c ii) are known as those of the degenerate supplementary series. The case $l=2$ in (c ii) was described in [1].

§6. The following theorem holds:

Theorem. *For a regular character χ, the representation $R(\chi)$ is unitary under the condition of* (i) *in* § 3 (*the non-degenerate principal series*) *and under the condition of* (ii) *in* § 3 (*the non-degenerate supplementary series*).

For a singular character χ of type (D), the factor representation $R(\chi)$ is unitary under the mixed conditions of (a i), (b), and (c i) in § 5 (*the degenerate principal series*) *and under the mixed conditions of (a ii), (b), and (c ii) in* § 5 (*the degenerate supplementary series*).

It is not known whether, for a general singular character χ, irreducible unitary representations (the aboves are all irreducible) are contained as subrepresentations or factor representations in $R(\chi)$. But these representations seem to us to be unitarily equivalent to those of the above types. In order to verify this conjecture, we need more detailed discussion about subrepresentations and factor representations of the non-unitary representation $R(\chi)$.

References

[1] E. M. Stein: Analysis in matrix spaces and some new representations of $SL(N, C)$. Ann. Math., **86**, 461–490 (1967).

[2] Gelfand-Graev-Vilenkin: Generalized Function. V (in Russian). Moscow (1962).

[3] M. Tsuchikawa: On the representations of $SL(3, C)$. I. Proc. Japan Acad., **43**, 852–855 (1967).

——: On the representations of $SL(3, C)$. II. Proc. Japan Acad., **44**, 127–129 (1968).

——: On the representations of $SL(3, C)$. III. Proc. Japan Acad., **44**, 130–132 (1968).