48. Smooth Structures on $S^p \times S^q$

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This paper shows the classification of smooth structures on $S^p \times S^q$ promised in [6].

In [10], Novikov classified smooth structures modulo one point of the manifolds which are tangentially homotopy equivalent to a product $S^p \times S^q$ of spheres. On the other hand, the author determined in his paper [6] the inertia group $I(S^p \times \tilde{S}^q)$ of $S^p \times \tilde{S}^q$. In the present paper, we shall show that combining these results derives complete classification of smooth structures on $S^p \times S^q$ for $p+q \geqq 6$, $1 \leqq p \leqq q$.

In the following we shall use the notations in [6].

Detailed proof will appear elsewhere.

1. Preliminaries. Let a smooth structure M_α on $S^p \times S^q$ be given i.e., assume that there is given a piecewise differentiable homeomorphism $f: S^p \times S^q \to M_\alpha$. Let x_0 (resp. y_0) denote a point of S^p (resp. S^q). Since $f(x_0 \times S^q)$ (resp. $f(S^p \times y_0)$) has a vector bundle neighbourhood in M_α, there exists a piecewise differentiable homeomorphism $h: M_\alpha \to M_\alpha$ such that $h(f(x_0 \times S^q))$ (resp. $h(f(S^p \times y_0))$) is a smooth submanifold of M_α (see R. Lashof and M. Rothenberg [9]). Therefore it follows that there exists a homotopy sphere $\tilde{S}^q$ (resp. $\tilde{S}^p$) which is embedded smoothly in M_α with a trivial normal bundle and which represents a generator of $H_q(M_\alpha) \cong H_q(S^p \times S^q) \cong Z$ (resp. $H_p(M_\alpha) \cong H_p(S^p \times S^q) \cong Z$) if $p \neq q$. We may assume that $\tilde{S}^p$ and $\tilde{S}^q$ intersect transversally at one point. Applying the similar argument as in [6], we can now show that

$$M_\alpha - \operatorname{Int} D^{p+q} = \tilde{S}^p \times D^q \,\forall\, D^p \times \tilde{S}^q = \tilde{S}^p \times \tilde{S}^q - \operatorname{Int} D^{p+q}$$

where $\forall$ denotes the plumbing of two manifolds. Hence M_α can be written as $\tilde{S}^p \times \tilde{S}^q \# \tilde{S}^{p+q}$ for some exotic sphere $\tilde{S}^{p+q}$, here $\#$ denotes the connected sum. It is easily seen that this still holds in the case $p=q$. Obviously $\tilde{S}^p \times \tilde{S}^q \# \tilde{S}^{p+q}$ is tangentially homotopy equivalent to $S^p \times S^q$. Therefore, by making use of the classification theorem of Novikov [10], we see that $\tilde{S}^p \times \tilde{S}^q$ is diffeomorphic to $S^p \times \tilde{S}^q$ modulo one point for $p \leqq q$. Thus the problem of classifying smooth structures on $S^p \times S^q$ ($p \leqq q$) is reduced to the study of smooth structures of the form $S^p \times \tilde{S}^q \# \tilde{S}^{p+q}$.

2. Lemmas. The following lemma is proved in Theorem C of [6].

Lemma 1. *Let $K_1: \pi_p(SO)\times\Theta_q \to \Theta_{p+q}$ denote the pairing defined by Milnor-Munkres-Novikov. Then it holds that*

$$I(S^p\times\tilde{S}^q)=K_1(\pi_p(SO),\tilde{S}^q)$$

for $p+q\geqq 5$, $p+1\neq q$.

The following lemma is a generalization of Corollary 3 of Katase [5].

Lemma 2. *$S^p\times\tilde{S}^q_1$ is diffeomorphic to $S^p\times\tilde{S}^q_2$ if $S^p\times\tilde{S}^q_1 \# \tilde{S}^{p+q}$ is diffeomorphic to $S^p\times\tilde{S}^q_2$ for some homotopy sphere $\tilde{S}^{p+q}$ for $p+q\geqq 5$ and $q\geqq p\geqq 1$.*

Define a subgroup G'_q of $G_q=\pi_{q+N}(S^N)$ (N: Large) as follows. A map $f: S^{q+N}\to S^N$ represents an element of G'_q if and only if f represents the Pontrjagin-Thom map of some framed imbedding $\tilde{S}^q\times D^N \subset S^{q+N}$.

Denote by $\mathcal{S}(M)$ the set of smooth structures on M modulo orientation preserving diffeomorphisms. Define that $M_\alpha, M_\beta\in\mathcal{S}(M)$ are equivalent if and only if there exists an orientation preserving diffeomorphism $f: M_\alpha\to M_\beta$ modulo one point. Denote by $\mathcal{S}'(M)$ the quotient set of $\mathcal{S}(M)$. The following is a revised form of the classification theorem of Novikov [10].

Lemma 3. *$\mathcal{S}'(S^p\times S^q)$ is in one-to-one correspondence with $G'_q/\sim$ for $p+q\geqq 6$ and $q\geqq p\geqq 2$ where $\sim$ is the relation of Novikov.*

3. Smooth structures on $S^p\times S^q$.

For $\alpha,\beta\in\Theta_q$, define $\alpha\sim\beta$ if and only if $\alpha=\beta$ or $\alpha=-\beta$. Denote by $\Theta_q/\sim$ the quotient set of Θ_q.

Theorem. *For $p+q\geqq 6$, $2\leqq p\leqq q$, we have*

$$\mathcal{S}(S^p\times S^q)=\{S^p\times\tilde{S}^q_i \# \tilde{S}^{p+q}_{ij} \mid \tilde{S}^q_i\in G'_q/\sim,\ \tilde{S}^{p+q}_{ij}\in\Theta_{p+q}/K_1(\pi_p(SO),\tilde{S}^q_i)\}.$$

For $p=1$ and $q\geqq 5$, we have

$$\mathcal{S}(S^1\times S^q)=\{S^1\times\tilde{S}^q_i \# \tilde{S}^{1+q}_{ij} \mid \tilde{S}^q_i\in\Theta_q/\sim,\ \tilde{S}^{1+q}_{ij}\in\Theta_{1+q}/K_1(\pi_1(SO),\tilde{S}^q_i)\}.$$

Combining lemmas in §2, we easily obtain this Theorem.

4. Some computations.

In this section we shall show some examples.

Proposition 1. *If (p,q) is any of the following: (2,7), (2,8), (6,8), (2,14), (3,13), (3,15), (6,10), then $\mathcal{S}(S^p\times S^q)=(G'_q/\sim)\times\Theta_{p+q}$.*

Proof. Bredon showed in [1] that if (p,q) is any of the set above, then $K_1(\pi_p(SO),\Theta_q)=S^{p+q}$ (the natural sphere). Therefore this is an immediate consequence of Theorem.

Proposition 2.

$$\mathcal{S}(S^3\times S^{10})=\{S^3\times S^{10},\ S^3\times S^{10}\#\tilde{S}^{13},\ S^3\times S^{10}\# 2\tilde{S}^{13},\ S^3\times\tilde{S}^{10}\},$$

i.e., $S^3\times S^{10}$ admits exactly 4 smooth structures, where $\tilde{S}^{10}$ denotes a generator of the three component Z_3 of $\Theta_{10}\cong Z_2\oplus Z_3$, and $\tilde{S}^{13}$ denotes a generator of $\Theta_{13}\cong Z_3$.

This follows from the following computations.

Kervaire and Milnor showed in [8] that every element of the group $G_{10} \cong Z_2 \oplus Z_3$ is represented by the Pontrjagin-Thom map of some framed imbedding $\tilde{S}^{10} \times D^N \subset S^{10+N}$. Since non-zero elements of the 3-component Z_3 of $G_{10} \cong Z_2 \oplus Z_3$ do not come from the unstable group $\pi_{14}(S^4)$ by the suspension homomorphism

$$E: \pi_{14}(S^4) \to \pi_{10+N}(S^N) \quad (N;\ \text{large}),$$

$S^3 \times \tilde{S}^{10}$ is not diffeomorphic to $S^3 \times S^{10}$ modulo one point for a generator $\tilde{S}^{10}$ of $Z_3 \subset Z_2 \oplus Z_3 \cong \Theta_{10} \cong G_{10} = G'_{10}$. In [6], it is shown that $I(S^3 \times \tilde{S}^{10}) = K_1(\pi_3(SO), \tilde{S}^{10}) = \Theta_{13}$.

On the other hand, the 2-component Z_2 of $G_{10} \cong Z_2 \oplus Z_3$ comes from the unstable group $\pi_{12}(S^2)$ (see H. Toda [11]). Therefore $S^3 \times \tilde{S}^{10}$ is diffeomorphic to $S^3 \times S^{10}$ modulo one point for the generator $\tilde{S}^{10}$ of the 2-component Z_2. By Lemma 2, we can deduce that $S^3 \times \tilde{S}^{10}$ is actually diffeomorphic to $S^3 \times S^{10}$. Therefore Theorem gives the requiring result.

Remark 1. Let $\tilde{S}^{10}$ denote a generator of the 3-component $Z_3 \subset G'_{10} = Z_2 \oplus Z_3$. Since there exist orientation reversing diffeomorphisms S^3 to S^3 and $\tilde{S}^{10}$ to $2\tilde{S}^{10}$ respectively, we have an orientation preserving diffeomorphism $f: S^3 \times \tilde{S}^{10} \to S^3 \times 2\tilde{S}^{10}$.

Proposition 3. *The order of* $\mathcal{S}(S^3 \times S^{14})$ *is* 24.

Proof. Since $G'_{14} = Z_2 \subset G_{14} = Z_2 \oplus Z_2$ (see Kervaire and Milnor [8]), $\mathcal{S}(S^3 \times S^{14})$ is the quotient set of $Z_2 \times \Theta_{14}$. In [6], we showed that $I(S^3 \times \tilde{S}^{14}) = K_1(\pi_3(SO), \tilde{S}^{14}) = Z_2 \not\equiv 0$ modulo $\Theta_{17}(\partial\pi)$ for the generator $\tilde{S}^{14}$ of $\Theta_{14} \cong Z_2$. Consequently we have

$$\mathcal{S}(S^3 \times S^{14}) = \{S^3 \times S^{14} \# \tilde{S}^{17}_i \mid \tilde{S}^{17}_i \in \Theta_{17}\}$$
$$\cup \{S^3 \times \tilde{S}^{14} \# \tilde{S}^{17}_j \mid \tilde{S}^{14} \neq S^{14},\ \tilde{S}^{17}_j \in \Theta_{17}/K_1(\pi_3(SO), \tilde{S}^{14})\}$$

which proves the proposition.

Proposition 4. *If* $p \leqq q \leqq p+3$, *then* $\mathcal{S}(S^p \times S^q)$ *is in one-to-one correspondence with* Θ_{p+q} *by* $S^p \times S^q \# \tilde{S}^{p+q} \mapsto \tilde{S}^{p+q}$.

Proof. Hsiang, Levine and Szczarba [4] shows that $\tilde{S}^q$ can be embedded in the $(q+p+1)$-dimensional euclidean space R^{q+p+1} with a trivial normal bundle for $q-2 \leqq p+1$. Therefore we can embed the natural sphere S^q in the disk bundle $B = \tilde{S}^q \times D^{p+1}$ such that S^q generates the q-dimensional homology group $H_q(B) \cong H_q(\tilde{S}^q \times D^{p+1}) \cong Z$ and that S^q has a trivial normal bundle (see Kervaire [7]). Making use of the Smale's h-cobordism theorem, it is easily verified that $\tilde{S}^q \times S^p$ is diffeomorphic to $S^q \times S^p$. It follows from Corollary 3 in [6] that the inertia group $I(S^p \times \tilde{S}^q)$ is trivial. Hence Theorem proves the proposition.

Proposition 5. *If* $p \equiv 2, 4, 5, 6 \pmod 8$, *then* $\mathcal{S}(S^p \times S^q)$ *is in one-to-one correspondence with* $(G'_q/\sim) \times \Theta_{p+q}$.

Proof. Since $K_1(\pi_p(SO), \tilde{S}^q) = K_1(0, \tilde{S}^q) = S^{p+q}$ for $p \equiv 2, 4, 5, 6$ (mod

8), the inertia group of $S^p \times \tilde{S}^q$ is trivial for every $\tilde{S}^q \in \Theta_q$. Therefore this proposition is an immediate consequence of Theorem.

5. A concluding remark.

Remark 2. The similar argument proves the classification theorem of smooth structures on a sphere bundle over sphere with a cross section. Denote by $M_h(\tilde{S}^q)$ the p-sphere bundle over a homotopy q-sphere $\tilde{S}^q$ with a characteristic map h which belongs to the image of the natural map $s: \pi_{q-1}(SO_p) \to \pi_{q-1}(SO_{p+1})$. Define a homomorphism $K[h, \tilde{S}^q]: \pi_p(SO_q) \to \Theta_{p+q}$ by

$$K[h, \tilde{S}^q](l) = K_1(l, \tilde{S}^q) + K_2(l, h)$$

(see Kawakubo [6]). *Then we have*

$$\mathcal{S}(M_h(S^q)) = \{M_h(\tilde{S}^q_i) \# \tilde{S}^{p+q}_{ij} \mid \tilde{S}^q_i \in G'_{(q)}/\sim,$$
$$\tilde{S}^{p+q}_{ij} \in \Theta_{p+q}/K[h, \tilde{S}^q_i](\pi_p(SO_q))\} \text{ for } p+q \geqq 6 \text{ and } q+2 \geqq p \geqq 2.$$

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