109. *A Non-Commutative Integration Theory for a Semi-Finite AW*-algebra and a Problem of Feldman*

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We shall extend Feldman's result on "Embedding of AW^*-algebras" to semi-finite AW^*-algebras, that is, we shall show that a semi-finite AW^*-algebra with a separating set of states which are completely additive on projections (c.a. states) has a faithful representation as a semi-finite von Neumann algebra. Full proofs will appear elsewhere.

Let M be a semi-finite AW^*-algebra with a separating set $\mathfrak{S}$ of c.a. states. By a c.a. state ϕ on M we mean a state on M such that for any orthogonal family of projections $\{e_i\}$ in M with $e=\sum_i e_i$ $\phi(e)=\sum_i \phi(e_i)$. Let C be the algebra of "measurable operators" affiliated with M [6]. Denote the set of all positive elements, projections, partial isometries and unitary elements in M by M^+, M_p, M_{pi} and M_u, respectively.

Let $\tilde{\mathfrak{S}}$ be the set of finite linear combinations of elements in $\{a^*\omega a, \omega \in \mathfrak{S}, a \in M\}$, where $(a^*\omega a)(x)=\omega(axa^*)$ for all $x \in M$. For any positive number ε and any positive integer n, put $V_{\varepsilon,n}(\omega_1, \omega_2, \cdots, \omega_n)(0) =\{a ; |\omega_i(a)|<\varepsilon, i=1,2,\cdots n, \omega_1, w_2, \cdots, \omega_n \in \tilde{\mathfrak{S}}\}$ and we define the $\sigma(\mathfrak{S})$-topology of M by assigning sets of the form $V_{\varepsilon,n}(\omega_1, \omega_2, \cdots, \omega_n)(0)$ to be its neighborhood system of 0. Since $\tilde{\mathfrak{S}}$ is a separating set of continuous linear functionals on M, this topology is the separated locally convex topology defined by the family of semi-norms $q_\omega(x)=|\omega(x)|$, $\omega \in \tilde{\mathfrak{S}}$. Then we have, by [3, Lemma 3],

Lemma 1. *Let $\{e_\alpha\}\alpha \in A$ be an orthogonal set of projections in M such that $e=\mathrm{Sup}\ [\sum\{e_\alpha, \alpha \in I\}$, $A \supset I \in F(A)$ where $F(A)$ is the family of all finite subsets of A], then $\sum\{e_\alpha, \alpha \in I\} \to e(I \in F(A))$ in the $\sigma(\mathfrak{S})$-topology.*

Lemma 2. *Any abelian AW^*-subalgebra, especially, the center Z of M is a W^*-algebra* ([7]) *and the $\sigma(\mathfrak{S})$-topology restricted to this subalgebra is equivalent to the σ-topology on bounded spheres.*

Let $\boldsymbol{Z}$ be the set of all $[0, +\infty]$-valued continuous functions on the spectrum of Z [1], then we have

Theorem 1. *There is an operation Φ from M^+ to $\boldsymbol{Z}$ having the following properties:*

(i) $\Phi(h_1+h_2)=\Phi(h_1)+\Phi(h_2)$ $h_1, h_2 \in M^+$;

(ii) $\Phi(\lambda h)=\lambda \Phi(h)$ *if λ is a positive number and $h \in M^+$*;

(iii) $\Phi(st)=t\cdot\Phi(s)$ $s \in M^+$, $t \in Z^+$;

(iv) $\Phi(uau^{-1})=\Phi(a)$ *if* $a \in M^+$ *and* $u \in M_u$;

(v) *for any* $a \in M^+$ *with* $\Phi(a)=0, a=0$;

(vi) *for every directed increasing net* $\{a_\alpha\}$ *in* M^+ *such that* $a_\alpha \to a$ *in the* $\sigma(\mathfrak{S})$*-topology for some* a *in* M, $\Phi(a_\alpha) \uparrow \Phi(a)$ *in* Z;

(vii) *for every non-zero* a *in* M^+, *there exists a nonzero* $b \in M^+$ *majorized by* a *such that* $\Phi(b) \in Z^+$.

Then by the above theorem and [3, Lemma 2], we have

Proposition 1. *In Theorem* 1, *let* $\mathcal{P}$ *be the set* $\{s \in M,\ s\geqq 0,\ \Phi(s) \in Z^+\}$, *then* $\mathcal{P}$ *is the positive part of a two-sided ideal* $\mathfrak{N}$ *and there exists a unique linear operation* $\dot{\Phi}$ *on* $\mathfrak{N}$ *to* Z *which coincides with* Φ *on* $\mathcal{P}$; *moreover this linear operation satisfies the following properties*;

(a) *If* $t \in \mathfrak{N}$ *with* $t\geqq 0$ *and* $\dot{\Phi}(t)=0$ *only if* $t=0$;

(b) $\dot{\Phi}(st)=\dot{\Phi}(ts)$ *if* $s \in M$ *and* $t \in \mathfrak{N}$;

(c) $\dot{\Phi}(st)=s\cdot\dot{\Phi}(t)$ *if* $s \in Z$ *and* $t \in \mathfrak{N}$;

(d) *let* $\{t_\mu\}$ *be a directed increasing net of positive elements in* $\mathfrak{N}$ *such that* $t_\mu \to t$ *in the* $\sigma(\mathfrak{S})$*-topology for some positive element* t *in* M *and if* $\{\dot{\Phi}(t_\mu)\}$ *is uniformly bounded, then* $t \in \mathfrak{N}$ *and* $\dot{\Phi}(t)=\mathrm{Sup}\{\dot{\Phi}(t_\mu), \mu\}$;

(e) *every non-negative element in* M *is the supremum of a set of non-negative elements in* $\mathfrak{N}$.

Now let p be a finite projection in M then there is an indexed family $\{e_\mu\}$ of mutually orthogonal central projections such that $\sum_\mu e_\mu =1$ and that for each μ $pMe_\mu p$ is a σ-finite finite AW^*-algebra. Therefore by Proposition 1(*e*), there is a sequence $\{p_n^{(\mu)}\}_{n=1}^\infty$ of mutually orthogonal projections in $\mathfrak{N}$ such that $pe_\mu=\sum_{n=1}^\infty p_n^{(\mu)}$. Write $D(p) =\sum_\mu \sum_{n=1}^\infty \dot{\Phi}(p_n^{(\mu)})$ in Z. If p is a properly infinite projection with central carrier $z(p)$, $D(p)(\omega)$ is defined as $\infty\cdot z(p)(\omega)$, thus we have

Theorem 2. *In* M, *we can define a dimension function* $D(e)$ *with values in* Z *for all projections* $e \in M$, *in such a way that*

(i) $D(e)(\omega)<\infty$ *except on a non-dense set if and only if* e *is finite*;

(ii) *if* $p, q \in M_p$ *and* $pq=0$, *then* $D(p+q)=D(p)+D(q)$;

(iii) *for any indexed chain of projections* $\{e_\lambda;\ \lambda \in \Lambda\}$ *in* M, $D(\bigvee_{\lambda\in\Lambda} e_\lambda) =\mathrm{Sup}\{D(e_\lambda), \lambda \in \Lambda\}$;

(iv) *if* u *is in* M_{pi}, *then* $D(u^*u)=D(uu^*)$;

(v) *for* $e \in Z_p$ *and* $p \in M_p$, $D(e)\neq 0$ *and* $D(ep)=eD(p)$.

Now along the same lines with [8], we introduce the notion of the "convergence nearly everywhere" of sequences in C.

Definition 1. We say that a sequence $\{x(n)\}_{n=1}^\infty$ of C converges nearly everywhere (or converges n.e.) to an element x in C if for any positive ε, there exist a positive integer $n_0(\varepsilon)$ and an SDD (strongly dense domain (see [6, Definition 3.1])) $\{e_n(\varepsilon)\}$ such that $(x(n)-x)[e_n(\varepsilon),1]$ $\varepsilon\ \bar{M}$ and $\|(x(n)-x)[e_n(\varepsilon), 1]\|_\infty<\varepsilon$ for all $n\geqq n_0(\varepsilon)$, where we write $\|[x, 1]\|_\infty=\|x\|$(see [6, Theorem 3.1, Lemma 5.2]).

Remark. (1) We must note that a limit nearly everywhere is unique. Making use of the dimension function (Theorem 2), by the same way as that of I. E. Segal, we have: (2) if $\{x(n)\}_{n=1}^\infty$ and $\{y(n)\}_{n=1}^\infty$ are sequences in C converging n.e. to x and y in C, respectively, then $\{x(n)+y(n)\}_{n=1}^\infty$ converges to $x+y$ n.e., (3) let $\{x(n)\}_{n=1}^\infty$ be a sequence in C which converges n.e. to x in C and suppose that there is a central projection e which is σ-finite with respect to the center such that $x(n)[1-e, 1]=0$ for all n, then there exists a strictly increasing subsequence $\{n_i\}$ of positive integers such that $\{x(n_i)^*\}_{n=1}^\infty$ converges n.e. to x^* and (4) in (3), for any y in C, there are subsequences $\{k_i\}$ and $\{m_i\}$ of positive integers such that $x(k_i)y\to xy(i\to\infty)$ and $yx(m_i)\to yx(i\to\infty)$ nearly everywhere.

Theorem 3. *There exists a* $[0, +\infty]$*-valued function* τ *(a faithful semi-finite trace) on* M^+ *having the following properties:*

(i) *If* $a, b\ \varepsilon\ M^+$, *then* $\tau(a+b)=\tau(a)+\tau(b)$;

(ii) *if* $a\ \varepsilon\ M^+$ *and* λ *is a positive number,* $\tau(\lambda a)=\lambda\tau(a)$ *(we recall here* $0\cdot+\infty=0$ *by our conventions);*

(iii) *if* $a\ \varepsilon\ M^+$ *and* $u\ \varepsilon\ M_u$, $\tau(u^*au)=\tau(a)$;

(iv) $\tau(a)=0$ $(a\ \varepsilon\ M^+)$ *implies* $a=0$;

(v) *for any non-zero* a *in* M^+, *there is a non-zero* b *in* M^+ *majorized by* a *such that* $\tau(b)<\infty$;

(vi) *let* $\{a_\alpha\}$ *be a directed increasing net of positive elements in* M *such that* $a_\alpha\to a$ *in the* $\sigma(\mathfrak{S})$*-topology for some* $a\ \varepsilon\ M$, *then* $\tau(a_\alpha)\uparrow\tau(a)$.

Then, there are a two-sided ideal $\mathcal{E}$, whose positive part is $\{a; a\ \varepsilon\ M^+, \tau(a)<\infty\}$ and a linear non-negative functional $\dot{\tau}$ on $\mathcal{E}$ coincides with τ on $\{a; a\ \varepsilon\ M^+, \tau(a)<\infty\}$ with the following properties:

(a) $\dot{\tau}(xy)=\dot{\tau}(yx)$ if x or $y\ \varepsilon\ \mathcal{E}$, x and $y\ \varepsilon\ M$,

(b) $\dot{\tau}(u^*xu)=\dot{\tau}(x)$ if $x\ \varepsilon\ \mathcal{E}$ and $u\ \varepsilon\ M_u$.

Let $\mathcal{F}$ be the set $\{a; a\ \varepsilon\ M, \tau(\mathrm{LP}(a))<\infty\}$ (where $\mathrm{LP}(a)$ is the left projection of a in M), then $\mathcal{F}$ is a two-sided ideal contained in $\mathcal{E}$ such that $\mathcal{E}_p=\mathcal{F}_p$.

Definition 2. An element x in C is integrable if there exists a sequence $\{x(n)\}_{n=1}^\infty$ in $\mathcal{F}$ such that $[x(n), 1]\to x$(n.e.) and $\dot{\tau}(|x(n)-x(m)|)\to 0$ as n and $m\to\infty$. The integral of x, in symbol $\bar{\tau}(x)$, is defined by $\bar{\tau}(x)=\lim_{n\to\infty}\dot{\tau}(x(n))$. The set of all integrable elements in C is denoted by $L^1(M, \tau)$.

Remark. Note first that the value $\bar{\tau}(x)$ of the integral of x in fact exists and is finite and that it is uniquely determined by any particular such sequences. Moreover by remark (2) following Definition 1, $\tilde{\tau}$ is linear on $L^1(M, \tau)$. Secondly we note that if $x \in \mathcal{E}$, then $[x, 1]$ is integrable and its integral is equal to $\dot{\tau}(x)$.

By the remark following Definition 1, we have

Proposition 2. (1) *For any* $s \in M$ *and* $t \in L^1(M, \tau)$, $[s, 1]t$, $t[s, 1]$ *and* $t^* \in L^1(M, \tau)$. *Moreover,* $\tilde{\tau}([s, 1]t) = \tilde{\tau}(t[s, 1])$ *and* $\tilde{\tau}(t^*) = \overline{\tilde{\tau}(t)}$ *(where* $\overline{\alpha}$ *is the complex conjugate of a complex number* α*).*

(2) *If* $p(\in M_p)$ *is integrable, then* $p \in \mathcal{E}_p$ *and* $\tilde{\tau}([p, 1]) = \dot{\tau}(p)$.

(3) *For any* $t \in L^1(M, \tau)$, *we define* $\|t\|_1 = \text{Sup}\{|\tilde{\tau}([s, 1]t)|, s \in M, \|s\| \leqq 1\}$. *Then the function* $t \to \|t\|_1 (t \in L^1(M, \tau))$ *satisfies actually the properties of a norm:*

(a) $0 \leqq \|t\|_1 < \infty$ *for* $t \in L^1(M, \tau)$ *and* $\|t\|_1 = 0$ *if and only if* $t = 0$,

(b) $\|s + t\|_1 \leqq \|s\|_1 + \|t\|_1$ *if* $s, t \in L^1(M, \tau)$,

(c) $\|\lambda t\|_1 = |\lambda| \cdot \|t\|_1$ *if* $t \in L^1(M, \tau)$ *and* λ *is a complex number,*

(d) $\|t\|_1 = \|t^*\|_1$,

(e) *if* $s \in M$, *then* $\|[s, 1]t\|_1 \leqq \|s\| \|t\|_1$ *and* $\|t[s, 1]\|_1 \leqq \|s\| \|t\|_1$.

(4) *The integral of a non-negative integrable element of* $\mathcal{C}$ *is non-negative.*

Definition 3. Let $L^2(M, \tau)$ be the set $\{t; t \in \mathcal{C}, t^*t = |t|^2 \in L^1(M, \tau)\}$ and we define $\|t\|_2 = \tilde{\tau}(|t|^2)^{1/2}$ for $t \in L^2(M, \tau)$.

Proposition 3. (1) *If* $s, t \in L^2(M, \tau)$, *then* $s^*t \in L^1(M, \tau)$ *and* $|\tilde{\tau}(s^*t)|^2 \leqq \|s\|_2^2 \|t\|_2^2$.

(2) $\|t\|_2 = \sup\{\|ts\|_1, \|s\|_2 \leqq 1, ts \in L^1(M, \tau)\}$ *(for* $t \in L^2(M, \tau)$*) and* $L^2(M, \tau)$ *is a pre-Hilbert space with respect to the norm* $\| \quad \|_2$. *Moreover this norm satisfies:*

(a) $\|t^*\|_2 = \|t\|_2 = \||t|\|_2$ *for* $t \in L^2(M, \tau)$,

(b) *for any* $s \in M$ *and* $t \in L^2(M, \tau)$, $[s, 1]t$ *and* $t[s, 1]$ *are in* $L^2(M, \tau)$. *Moreover* $\|[s, 1]t\|_2 \leqq \|s\| \|t\|_2$ *and* $\|t[s, 1]\|_2 \leqq \|s\| \|t\|_2$.

Theorem 4. $\overline{\mathcal{F}}(= \{[x, 1], x \in \mathcal{F}\})$ *is norm-dense in* $L^2(M, \tau)$ *and* $L^1(M, \tau)$, *respectively. Moreover* $L^1(M, \tau)$ *(resp.* $L^2(M, \tau)$*) is a Banach space with respect to the norm* $\| \quad \|_1$ *(resp.* $\| \quad \|_2$*). In particular,* $L^2(M, \tau)$ *is a Hilbert space.*

Now let us consider the left regular representation of M, which is defined by $\pi_l(x)a = [x, 1]a$, $a \in L^2(M, \tau)$, $x \in M$. Then by Proposition 3, $\pi_l(x)$ is a bounded linear operator on $L^2(M, \tau)$ for each $x \in M$. On the other hand $\pi_l(x) = 0$, then $[x, 1]a = 0$ for all $a \in L^2(M, \tau)$. Since τ is semi-finite, there is an orthogonal set $\{e(\alpha)\}$ of projections in $\mathcal{F}$ such that $\sum e(\alpha) = 1$. Therefore $\overline{\mathcal{F}} \subset L^2(M, \tau)$ implies that $xe(\alpha) = 0$ for all α. Hence by [4, Lemma 2.2], $x = 0$. Therefore $\pi_l(\cdot)$ is a $*$-isomorphism of M into $\boldsymbol{B}(L^2(M, \tau))$ (where $\boldsymbol{B}(L^2(M, \tau))$ is the algebra of all bounded linear operators on $L^2(M, \tau)$).

Let $\{g_i\}_{i \in I}$ be a set of mutually orthogonal projections of M with $e = \sum_{i \in I} g_i$, then for each $a \in \mathcal{F}$

$$\|\pi_l(e)[a, 1]-\sum_{i\in J}\pi_l(g_i)[a, 1]\|_2^2$$
$$=\dot{\tau}(a^*(e-\sum_{i\in J}g_i)a)$$

for any finite subset J of I. Therefore by Theorem 3 (v) and Theorem 4, $\sum_{i\in J}\pi_l(g_i)\to\pi_l(e)$ strongly. Thus $\pi_l(M)$ is an AW^*-subalgebra of $\boldsymbol{B}(L^2(M, \tau))$ in the sense of [5.3, Definition].

Let $\boldsymbol{M}$ be the weak closure of $\pi_l(M)$, then $\boldsymbol{M}$ is a von Neumann algebra on $L^2(M, \tau)$.

Theorem 5. *$\pi_l(M)=\boldsymbol{M}$, that is, M is a semi-finite W^*-algebra.*

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