141. On Some Results Involving Jacobi Polynomials and the Generalized Function $\tilde{\omega}_{\mu_1,\cdots,\mu_n}(x)$

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Abstract. The object of this paper is to evaluate the following type of multiple integrals:

$$\prod_{r=1}^{m}\int_0^1 x_r^{\rho_r}(1-x_r)^{\beta_r}P_{n_r}^{(\alpha_r,\beta_r)}(1-2x_r)dx_r\tilde{\omega}_{\mu_1,\cdots,\mu_n}[\lambda(x_1\cdots x_m)^{\pm h/2}].$$

These integrals are then employed to establish the expansions for the $\tilde{\omega}_{\mu_1,\cdots,\mu_n}(x)$ function involving Jacobi polynomials.

1. Introduction. The function $\tilde{\omega}_{\mu_1,\cdots,\mu_n}(x)$ was defined [1] by the integral equation

$$(1.1)\qquad \tilde{\omega}_{\mu_1,\cdots,\mu_n}(x)=x^{1/2}\int_0^\infty\cdots\int_0^\infty J_{\mu_1}(t_1)\cdots J_{\mu_{n-1}}(t_{n-1})J_{\mu_n}\left(\frac{x}{t_1\cdots t_{n-1}}\right)\cdot(t_1\cdots t_{n-1})^{-1}dt_1\cdots dt_{n-1},$$

$$=\int_0^\infty \tilde{\omega}_{\mu_1,\cdots,\mu_{n-1}}(x/t)J_{\mu_n}(t)t^{-1/2}dt$$

Where $R\left(\mu_k+\frac{1}{2}\right)\geq 0, k=1,2,\cdots,n$ and μ's may be permuted among themselves.

The following results are known.

(1.2) $\tilde{\omega}_\mu(x)=\sqrt{x}J_\mu(x),\quad \tilde{\omega}_{\mu,\mu+1}(x)=J_{2\mu+1}(2\sqrt{x}),\quad R(\mu)>-1.$

(1.3) The Mellin transform of $\tilde{\omega}_{\mu_1,\cdots,\mu_n}(x)$ is

$$2^{n(s-1/2)}\cdot\frac{\Gamma\left(\frac{\mu_1}{2}+\frac{s}{2}+\frac{1}{4}\right)\cdots\Gamma\left(\frac{\mu_n}{2}+\frac{s}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu_1}{2}-\frac{s}{2}+\frac{3}{4}\right)\cdots\Gamma\left(\frac{\mu_n}{2}-\frac{s}{2}+\frac{3}{4}\right)}.$$

In this paper we have evaluated some multiple integrals involving the above generalized function and empolyed them to obtain some expansion formulae for the generalized function $\tilde{\omega}_{\mu_1,\cdots,\mu_n}(x)$. Particular cases have also been given with proper choice of parameters.

2. The multiple integrals. The integrals to be evaluated are:

$$(2.1)\qquad \prod_{r=1}^{m}\int_0^1 x_r^{\rho_r}(1-x_r)^{\beta_r}P_{n_r}^{(\alpha_r,\beta_r)}(1-2x_r)dx_r\tilde{\omega}_{\mu_1,\cdots,\mu_n}[\lambda(x_1\cdots x_m)^{\pm h/2}]$$

$$=\frac{h^{-\Sigma\beta_r-1}}{\pi 2^{n/2}}\prod_{r=1}^{m}\left(\frac{\Gamma(\beta_r+n_r+1)}{\Gamma(n_r+1)}\right)\sum_{i,-i}\frac{1}{i}G_{2n+2mh+1,2mh+1}^{mh+1,mh+n+1}$$

$$\times\left(\frac{2^{2n}e^{i\pi}}{\lambda^2}\left|\begin{matrix}\left(\frac{3}{4}-\frac{\mu_j}{2}\right)_n, \Delta(h,\rho_j-\alpha_j-n_j+1)_m, 1,\\ \Delta(h,\rho_j+1)_m, 1,\end{matrix}\right.\right.$$

$$\left(\frac{3}{4}+\frac{\mu_j}{2}\right)_n, \Delta(h, \beta_j+\rho_j+n_j+2)_m \Big), \quad \Delta(h, \rho_j-\alpha_j+1)_m$$

where $R(\mu_k) \geq -\frac{1}{2}, k=1,2,\cdots,n, R(\rho,\beta)>-1$ and

(i) the symbol $\sum_{i,-i}$ means that in the expression following it, i is to be replaced by $-i$ and the two expressions are to be added.

(ii) The symbol $\left(\frac{3}{4}-\frac{\mu_j}{2}\right)_n$ denotes n-parameters $\frac{3}{4}-\frac{\mu_1}{2}, \frac{3}{4}-\frac{\mu_2}{2}, \cdots, \frac{3}{4}-\frac{\mu_n}{2}$.

(iii) the symbol $\Delta(h,\alpha)$ denotes h-parameters $\frac{\alpha}{h}, \frac{\alpha+1}{h}, \cdots, \frac{\alpha+h-1}{h}$ and $\Delta(h,\alpha_j)_m$ denotes mh-parameters:

$$\Delta(h,\alpha_1), \Delta(h,\alpha_2), \cdots, \Delta(h,\alpha_m).$$

(iv) h is a positive number.

(2.2)
$$\prod_{r=1}^{m}\int_0^1 x_r^{\rho_r}(1-x_r)^{\beta_r}P_{n_r}^{(\alpha_r,\beta_r)}(1-2x_r)dx_r\,\tilde{\omega}_{\mu_1,\dots,\mu_n}[\lambda(x_1\cdots x_m)^{-h/2}]$$
$$=\frac{h^{-\Sigma\beta_r-1}\prod_{r=1}^{m}\Gamma(\beta_r+n_r+1)}{\pi 2^{n/2}\prod_{r=1}^{m}\Gamma(n_r+1)}\sum_{i,-i}G^{mh+1,mh+n+1}_{2mh+2n+1,2mh+1}\left(\frac{e^{i\pi}2^{2n}}{\lambda^2}\middle|\begin{matrix}\left(\frac{3}{4}-\frac{\mu_j}{2}\right)_n, \Delta(h,-\rho_j)_m, 1, \left(\frac{3}{4}+\frac{\mu_j}{2}\right)_n, \Delta(h,\alpha_j-\rho_j)_m\\ \Delta(h,\alpha_j-\rho_j+n_j)_m, 1, \quad \Delta(h,-\beta_j-\rho_j-n_j-1)_m\end{matrix}\right),$$

where h is a positive number, $R(\rho_r,\beta_r)>-1$, $R\left(\mu_k+\frac{1}{2}\right)\geq 0$ and $\left(\frac{3}{4}-\frac{\mu_j}{2}\right)_n$, $\Delta(h,\rho_j)_m$ have the same meaning as before.

Proof. To prove (2.1), apply (1.3) to replace

$$\tilde{\omega}_{\mu_1,\dots,\mu_n}[\lambda(x_1\cdots x_m)^{+h/2}]$$

on the left of (2.1) by

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}2^{n(s-1/2)}\frac{\Gamma\left(\frac{\mu_1+s}{2}+\frac{1}{4}\right)\cdots\Gamma\left(\frac{\mu_n+s}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\mu_1-s}{2}+\frac{3}{4}\right)\cdots\Gamma\left(\frac{\mu_n-s}{2}+\frac{3}{4}\right)}[\lambda(x_1\cdots x_m)^{+h/2-s}]ds.$$

Then, on changing the order of integration and evaluating inner integral by means of ([2], p. 284), the integral becomes

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} 2^{n(s-1/2)} \frac{\prod_{j=1}^{n}\Gamma\left(\frac{\mu_j+s}{2}+\frac{1}{4}\right)}{\prod_{j=1}^{n}\Gamma\left(\frac{\mu_j-s}{2}+\frac{3}{4}\right)}\lambda^{-s}$$

$$\times\prod_{r=1}^{m}\left\{\frac{\Gamma\left(\rho_r+1-\frac{hs}{2}\right)\Gamma(\beta_r+n_r+1)\Gamma\left(\alpha_r-\rho_r+n_r+\frac{hs}{2}\right)}{n_r!\Gamma\left(\alpha_r+\frac{hs}{2}\right)\Gamma\left(\beta_r+\rho_r+n_r+2-\frac{hs}{2}\right)}\right\}ds$$

$$=\frac{2^{1-n/2}}{2\pi i}\int_{C_1}\frac{\prod_{j=1}^{n}\Gamma\left(\frac{\mu_j+2s}{2}+\frac{1}{4}\right)\prod_{r=1}^{m}}{\prod_{j=1}^{n}\Gamma\left(\frac{\mu_j-2s}{2}+\frac{3}{4}\right)\prod_{r=1}^{m}}$$

$$\times\frac{\{\Gamma(\rho_r+1-hs)\Gamma(\beta_r+n_r+1)\Gamma(\alpha_r-\rho_r+n_r+hs)\}}{\{\Gamma(n_r+1)\Gamma(\alpha_r-\rho_r+hs)\Gamma(\beta_r+\rho_r+n_r+2-hs)\}}\left(\frac{2^{2n}}{\lambda^2}\right)^s$$

$$\cdot\left\{\frac{e^{i\pi s}-e^{-i\pi s}}{2\pi i}\right\}\Gamma(s)\Gamma(1-s)ds,$$

where we have used the relation

$$\Gamma(\xi)\Gamma(1-\xi)=\frac{\pi}{\sin\pi\xi}.$$

Now apply ([3], p. 4 (11) and p. 207 [1]) to evaluate the integral and so obtain (2.1).

(2.2) can be proved by proceeding on similar lines.

3. The expansions. The expansions to be established are

(3.1) $x^{\rho}\tilde{\omega}_{\mu_1,\cdots,\mu_n}[\lambda x^{h/2}]$

$$=\frac{h^{-\beta-1}}{\pi 2^{n/2}}\sum_{r=0}^{\infty}\frac{(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+1)}{\Gamma(\alpha+r+1)}\sum_{i,-i}\frac{1}{i}G_{2h+2h+1,2h+1}^{h+1,h+n+1}$$

$$\times\left(\frac{2^{2n}e^{i\pi}}{\lambda^2}\middle|\begin{matrix}\left(\frac{3}{4}-\frac{\mu_j}{2}\right)_n, \Delta(h,\rho-r+1),1,\\ \Delta(h,\rho+\alpha+1),1,\end{matrix}\right.$$

$$\left.\times\begin{matrix}\left(\frac{3}{4}+\frac{\mu_j}{2}\right)_n, \Delta(h,\beta+\rho+\alpha+r+2)\\ \Delta(h,\rho+1)\end{matrix}\right)P_r^{(\alpha,\beta)}(1-2x),$$

where h is a positive number and $R(\rho)>-1$.

(3.2) $x^{\rho}\tilde{\omega}_{\mu_1,\cdots,\mu_n}[\lambda x^{-h/2}]$

$$=\frac{h^{-\beta-1}}{\pi 2^{n/2}}\sum_{r=0}^{\infty}\frac{(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+1)}{\Gamma(\alpha+r+1)}\sum_{i,-i}\frac{1}{i}G_{2h+2n+1,2h+1}^{h+1,h+n+1}$$

$$\times\left(\frac{e^{i\pi}2^{2n}}{\lambda^2}\middle|\begin{matrix}\left(\frac{3}{4}-\frac{\mu_j}{2}\right)_n, \Delta(h,\alpha-\rho),1,\\ \Delta(h,r-\rho),1,\end{matrix}\right.$$

$$\left.\begin{matrix}\left(\frac{3}{4}+\frac{\mu_j}{2}\right)_n, \Delta(h,-\rho)\\ \Delta(h,-\alpha-\beta-\rho-r-1)\end{matrix}\right)P_r^{(\alpha,\beta)}(1-2x),$$

where h is a positive number and $R(\rho)>1$.

Proof. To prove (3.1), let

$$f(x)=x^{\rho}\tilde{\omega}_{\mu_1,\cdots,\mu_n}[x^{h/2}\lambda] = \sum_{r=0}^{\infty} C_r P_r^{(\alpha,\beta)}(1-2x). \tag{3.3}$$

Equation (3.3) is valid, since $f(x)$ is continuous and of bounded variation in the interval $(0,1)$ when $R(\rho)\geq -1$.

Multiplying both sides of (3.3) by $x^{\alpha}(1-x)^{\beta}P_u^{(\alpha,\beta)}(1-2x)$ and integrating with respect to x from 0 to 1, we get

$$\int_0^1 x^{\rho+\alpha}(1-x)^{\beta}P_u^{(\alpha,\beta)}(1-2x)\tilde{\omega}_{\mu_1,\cdots,\mu_n}[\lambda x^{h/2}]dx = \sum_{r=0}^{\infty} C_r \int_0^1 x^{\alpha}(1-x)^{\beta}P_u^{(\alpha,\beta)}(1-2x)P_r^{(\alpha,\beta)}(1-2x)dx.$$

Now using (2.1) and the orthogonality property of Jacobi polynomials ([2], p. 285 (9) and (10)), we get

$$C_u=\frac{h^{-\beta-1}(\alpha+\beta+2u+1)\Gamma(\alpha+\beta+u+1)}{\pi 2^{n/2}\Gamma(\alpha+u+1)} \sum_{i,-1} \frac{1}{i} G^{h+1,h+n+1}_{2n+2h+1,2h+1} \times\left(\frac{2^{2n}e^{i\pi}}{\lambda^2}\middle| \begin{matrix}\left(\frac{3}{4}-\frac{\mu_j}{2}\right)_n, \Delta(h,\rho-u+1), 1, \left(\frac{3}{4}+\frac{\mu_j}{2}\right)_n, \Delta(h,\beta+\rho+\alpha+u+2) \\ \Delta(h,\rho+\alpha+1), 1, \Delta(h,\rho+1)\end{matrix}\right). \tag{3.4}$$

From (3.3) and (3.4), the formula (3.1) is obtained. The expansion formula (3.2) is similarly established on applying the same procedure as above and using (2.2).

References

[1] Bhatnagar, K. P.: Two theorems on self-reciprocal functions and a new transform. Bull. Calcutta Math. Soc., **45**, 109–112 (1953).
[2] Erdelyi, Magnus: Oberhetinger, Tricomi. Tables of Integral Transforms, Vol. II. McGraw-Hill, Bateman project (1954).
[3] ——: Higher Transcendental Functions, Vol. I. McGraw-Hill, Bateman project (1953).