138. *Rings with an almost Nil Adjoint Semigroup*

By Ferenc A. SZÁSZ
Budapest

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Throughout this note ring means always an associative ring. The used fundamental notions can be found in [1] and [3]. As it is well known, the elements of a ring A form with respect to the circle operation $x\circ y = x+y-xy$ a semigroup S, which is called the adjoint semigroup of the ring A. By $x\circ 0 = 0\circ x = x$ for any $x\in A$, the zero element 0 of A is the twosided unity element of S. Furthermore $e\in A$ is the zero element of the adjoint semigroup S if and only if e is the twosided unity element of the ring A, being then $x\circ e = e\circ x = e$ for any $x\in A$. For results on the circle operation we refer yet to [2]–[4] and [6].

Let S be an arbitrary semigroup, having both twosided unity element and zero. Then S is said to be almost nil, if any element of S, which is different from the twosided unity element of S, is nilpotent. Therefore the nil radical of an arbitrary almost nil semigroup can be very big. Some results on general radicals of semigroups with zero element are discussed in [5].

The aim of this note is to determine all rings A having almost nil adjoint semigroups S. Any here discussed ring A must have twosided unity element e, which is the zero element of the almost nil adjoint semigroup S. Instead of "almost nil" obviously cannot be taken "nil", because S has twosided unity element, which is the zero element 0 of A.

Any nontrivial ring having an almost nil adjoint semigroup can be considered as a very strongly nonradical ring for the Jacobson radical [3], being the adjoint semigroup of any Jacobson radical ring a group.

Proposition 1. *For any nonzero element a of a ring A with an almost nil adjoint semigroup S there exists a natural number n such that $(e-a)^n=0$ holds, where e is the twosided unity element of A.*

Proof. We have obviously $x\circ y = e-(e-x)\cdot(e-y)$ for any $x, y\in A$. Consequently for any nonzero element a of A (namely for any element a of S, which is different from the unity element of S) there exists a natural number n such that $e-(e-a)^n = e$, being e the zero of the almost nil adjoint semigroup S. Therefore $(e-a)^n=0$ holds.

Proposition 2. *Any nonzero element a of a ring A with an almost nil adjoint semigroup S cannot be quasiregular in the sense of*

Jacobson [3].

Proof. Assume $0 \neq a \in A$ and $a+b-ab=a+b-ba=0$ with some $b \in A$. Then for the twosided unity element e of A we have $e=(e-a)\cdot(e-b)=(e-b)(e-a)$. On the other side Proposition 1 yields at once $(e-a)^n=0$ for some n, and thus the obtained contradiction

$$e=e^n=((e-a)(e-b))^n=(e-a)^n\cdot(e-b)^n=0\cdot(e-b)^n=0$$

completes the proof.

Theorem. *A ring A has an almost nil adjoint semigroup S if and only if A is a field with only two elements.*

Proof. Obviously the field with only two elements has an almost nil adjoint semigroup of order two. Conversely, it is sufficient to verify, that if the adjoint semigroup S of the ring A is almost nil, then any nonzero element a of A coincides with the twosided unity element e of A. By Proposition 1 we have $(e-a)^n=0$ for any nonzero $a \in A$ with some n. Therefore the element $a-e$ is nilpotent, and thus $a-e$ is also quasiregular in the sense of Jacobson [3]. Consequently, by Proposition 2 it follows $a-e=0$ and $a=e$, which completes the proof.

References

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