177. *Topological Groups and the Generalized Continuum Hypothesis*

By Ofelia Teresa ALAS
Institute of Mathematics, University of São Paulo, São Paulo, Brazil

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

The purpose of this Note is to show that the proposition (I) and (II) below are equivalent to the Generalized Continuum Hypothesis, in the Zermelo-Fraenkel set theory, without the Axiom of Choice. Propositions (I) and (II) deal with a topological property—property K —of a particular type of topological group. Property K is related with the uniform continuity of all continuous real-valued functions defined on a group.

1. Preliminaries and notations.

We consider the Zermelo-Fraenkel axiomatic set theory, without the Axiom of Choice.

For any set $Z, 2^Z$ is the potence set of Z and $2Z$ is the set $\{0,1\}\times Z$. In §3, according to the notations of [2], for any set Z, we put $P_0(Z)=Z$, $P(Z)=2^Z$ and $P_i(Z)=P(P_{i-1}(Z))$, $i=1,2,3,4$.

For any two sets A and $B, A\leqslant B$ means that there is an injective map from A into B; $A<B$ means that $A\leqslant B$ and A and B are not equipotent sets; $A\approx B$ means that A and B are equipotent sets (i.e., by virtue of Bernstein-Cantor theorem, $A\leqslant B$ and $B\leqslant A$). Finally, $A+B$ indicates the disjoint sum of the sets A and B.

Let (G,τ) be a topological group and let U denote the right uniformity of G. (G,τ) *has property* K if and only if any continuous real-valued function on G is a uniformly continuous map of (G,U) into R (i.e., if $f: G\to R$ is continuous and r is a positive real number, there is an open neighborhood of the neutral element of G, V, such that if $x,y\in G$ and $y\in Vx$, then $|f(x)-f(y)|<r$). The group operation is denoted multiplicatively.

Let S be an infinite set. $\{0,1\}^S$ is an algebraic group with the following operation: if $x=(x_s)_{s\in S}$ and $y=(y_s)_{s\in S}$ belong to $\{0,1\}^S$, then $xy=(x_sy_s)_{s\in S}$, where $01=10=1$ and $00=11=0$.

Let M be an infinite set, S be a set with $2^M\leqslant S$, and put $G(S)=\{x\in\{0,1\}^S \mid \{s\in S \mid x_s=1\}\leqslant M\}$, where $x=(x_s)_{s\in S}$. We say that $G(S)$ is a group if $G(S)$ is a subgroup of $\{0,1\}^S$.

For any infinite sets M, Y and S, with $M<Y\leqslant 2^M\leqslant S$, let $B(Y)$ be the set of all elements $G(S)\cap\prod_{s\in S}V_s$, where $V_s\subset\{0,1\}$, $\forall s\in S$ and

$\{s \in S \mid V_s \neq \{0,1\}\} \leqslant Y$. We shall denote by $\tau(Y)$ the set of all unions of elements of $B(Y)$.

Remark. If $G(S)$ is a group and $\tau(Y)$ is a topology in $G(S)$, then $G(S)$ with the topology $\tau(Y)$ is a Hausdorff topological group. Thus, denoting by $G(S,Y)$ the pair $(G(S),\tau(Y))$, *$G(S,Y)$ is a topological group* means that $G(S)$ is a group and $\tau(Y)$ is a topology in $G(S)$.

2. Propositions (I) and (II).

(I) *For any infinite sets M, Y and S, with $M < Y \leqslant 2^M \leqslant S$, if $G(S,Y)$ is a topological group, then $G(S,Y)$ has property K.*

(II) *For any infinite sets M and Y, with $M < Y \leqslant 2^M$, if $G(2^M, Y)$ is a topological group, then $G(2^M, Y)$ has property K.*

It is obvious that (I) implies (II).

Theorem 1. *For any infinite sets M, Y and S, with $M < Y \leqslant 2^M \leqslant S$, if $G(S,Y)$ is a topological group and has property K, then $Y \approx 2^M$.*

Proof. On the contrary, let us suppose that there are infinite sets M, Y and S, with $M < Y < 2^M \leqslant S$, such that $G(S,Y)$ is a topological group and has property K.

The set 2^M is equipotent with a subset of S; to simplify the notations, we shall identify both sets; thus we shall suppose that $2^M \subset S$.

For any $P \in 2^M$, let $V(P)$ be equal to the set $\prod_{s \in S} V_s$, where $V_P = \{0\}$ and $V_s = \{0,1\}$ otherwise. It follows that the set $\bigcap_{P \in 2^M} G(S) \cap V(P)$ *is not* a neighborhood of the neutral element of $G(S)$.

Putting $M' = \{\{t\} \mid t \in M\}$ (thus $M' \subset S$), for any $P \in 2^M$, let a_p be the element $(x_s)_{s \in S}$, where $x_s = 1$ if $s \in \{\{t\} \mid t \in P\}$ and $x_s = 0$ otherwise. Finally, put $W = \prod_{s \in S} W_s$, where $W_s = \{0\}$ if $s \in M'$ and $W_s = \{0,1\}$ otherwise.

Now, we consider the open-closed set A, where $A = \bigcup_{P \in 2^M} (G(S) \cap V(P) \cap W) a_P$. Since $G(S,Y)$ has property K, there exists a set $V \in B(Y)$, neighborhood of the neutral element of $G(S)$, such that $V \subset W$ and $VA = A$. It follows that $V \subset \bigcap_{P \in 2^M} G(S) \cap V(P)$, which is impossible. So we have $Y \approx 2^M$.

Corollary. (II) *implies that for any infinite sets M and Y, with $M < Y \leqslant 2^M$, if $G(2^M, Y)$ is a topological group, then $Y \approx 2^M$.*

3. In this section we shall prove that (II) implies the Generalized Continuum Hypothesis.

Generalized Continuum Hypothesis: *For any infinite set M, there is no set Z, such that $M < Z < 2^M$.*

Zermelo theorem: *Any set can be well-ordered.*

Theorem 2. (II) *implies the Zermelo theorem.*

To prove Theorem 2 we shall need the following two lemmas ([2], pages 148-149):

Lemma 1. *For any infinite set X, there is a well-ordered set W, such that $W \subset P_4(X)$ and we do not have $W \leqslant X$.*

Lemma 2. *If X and Y are sets such that $X+Y \approx P(2X)$, then $Y \geqslant P(X)$.*

Remark. Let A be a nonempty set and let N be the set of all natural numbers. Put $B=P(A+N)$; thus, since $A+N+N \approx A+N$, we have $N \times B \leqslant 2^N \times 2^{A+N} \approx 2^{A+N+N} \approx B$. For any $i=1,2,3,4$, we have that $P_i(B) \leqslant 2P_i(B) \leqslant N \times P_i(B) \leqslant 2^N \times 2^{P_{i-1}(B)} \leqslant 2^{N \times P_{i-1}(B)} \approx P_i(B)$. The set A is naturally embedded in B. If B can be well-ordered, then A can be well-ordered.

The proof of Theorem 2 follows that of Cohen [2], page 149.

Proof of Theorem 2.

Let X be a set; we shall prove that X can be well-ordered. We may suppose that $X \neq \emptyset$ (if X is empty the result is obvious). First, we suppose that $N \times P_i(X) \approx P_i(X)$ for $i=1,2,3,4$ (by virtue of the remark above, it is sufficient to consider this case). Let W be a well-ordered set, such that $W \subset P_4(X)$ and we do not have $W \leqslant X$.

Since $N \times P_4(X) \approx P_4(X)$, we have $P_3(X) \leqslant N \times W + P_3(X) \leqslant P_4(X)$. By virtue of Theorem 1, it is not possible that $P_3(X) < N \times W + P_3(X) < P_4(X)$, because $G(P_4(X), Y)$, where $M=P_3(X)$ and $Y=N \times W+P_3(X)$, has property K. Thus, we have $N \times W \leqslant P_3(X)$ or $N \times W \geqslant P_4(X)$, by virtue of Lemma 2. In this last case, $P_4(X)$ and, thus, X, can be well-ordered, because $N \times W$ can be well-ordered.

If $N \times W \leqslant P_3(X)$, we have $P_2(X) \leqslant N \times W + P_2(X) \leqslant P_3(X)$. So we may use the same argument as before. It follows that $N \times W \leqslant P_2(X)$ or $N \times W \geqslant P_3(X)$. (We apply the same argument again.)

Finally, we recall that if $X \leqslant N \times W + X \leqslant P_1(X)$, then, necessarily, we have $N \times W \geqslant P_1(X)$. The proof is completed.

Theorem 3. (II) *is equivalent to the* GCH.

Proof. Let M and Y be two infinite sets, with $M < Y \leqslant 2^M$. Since (II) implies the Zermelo theorem, and thus the Axiom of Choice, we have $M \approx 2M$ and $Y \approx 2Y$. It follows that $G(2^M)$ is a group and $\tau(Y)$ is a topology in $G(2^M)$. Thus, by the corollary of Theorem 1, $Y \approx 2^M$.

Now, let us prove that the GCH implies (II). Let M and Y be two infinite sets, with $M < Y \leqslant 2^M$. By the GCH, $Y \approx 2^M$ and, since the GCH implies the Zermelo theorem ([2], page 149), $G(2^M)$ is a group. It is obvious that $\tau(Y)$ is the discrete topology in $G(2^M)$; thus, $G(2^M, Y)$ has property K.

4. In the present section, we suppose that the Generalized Continuum Hypothesis (and, thus, the Axiom of Choice) is verified. It follows that for any infinite sets M and S, with $2^M \leqslant S, G(S, 2^M)$ is a topological group.

Theorem 4. *For any infinite sets M and S, with $2^M \leqslant S, G(S, 2^M)$ has property K.*

Proof. Let M and S be two infinite sets, with $2^M \leqslant S$. If $S \approx 2^M$, then $G(S, 2^M)$ is a discrete topological group and the result is obvious. So let us suppose that $2^M < S$. To prove that $G(S, 2^M)$ has property K is sufficient to show that for any open-closed subset A of $G(S)$, there is a neighborhood W of the neutral element of $G(S)$, such that $WA = A$.

To simplify the notations, we denote by D the set $\{V \subset \{0,1\}^S \mid \{s \in S \mid V_s \neq \{0,1\}\} \leqslant 2^M\}$, where $V = \prod_{s \in S} V_s$. Let μ be the first ordinal number of cardinality equal to $2^{|M|}$ and H be the set of all ordinal numbers smaller than μ. If α and β belong to $H, \alpha \ll \beta$ means that the ordinal number α is smaller than the ordinal number β. (We use $\ll$ to avoid any confusion with $<$, introduced at § 1.) We recall that if $\alpha \in H$, then α', the ordinal successor of α, belongs to H.

Let A be an open-closed subset of $G(S), \emptyset \neq A \neq G(S)$, and B be the complementary set of A; thus, $G(S) = A \cup B$.

Let us suppose that there exists a family $(S_\alpha)_{\alpha \in H}$ of subsets of S, verifying the following conditions:

a) $S_\alpha \leqslant 2^M, \forall \alpha \in H$ and $S_\beta \subset S_\alpha, \forall \alpha, \beta \in H, \beta \ll H\alpha$;

b) if $x \in G(S), x = (x_s)_{s \in S}$, and $\{s \in S \mid x_s = 1\} \subset S_\alpha$, for a convenient $\alpha \in H$, then there exists a set $V \in D$, $V = \prod_{s \in S} V_s$, such that $x \in V$, $\{s \in S \mid V_s \neq \{0,1\}\} \subset S_{\alpha'}$, where α' is the ordinal successor of α, and $G(S) \cap V \subset A$ or $G(S) \cap V \subset B$.

Putting $P = \bigcup_{\alpha \in H} S_\alpha$ and $W = \prod_{s \in S} W_s$, where $W_s = \{0\}$ if $s \in P$ and $W_s = \{0,1\}$ otherwise, it follows that $(G(S) \cap W)A = A$. Indeed, if $(a_s)_{s \in S}$ belongs to A, there is $\alpha \in H$, such that $\{s \in P \mid a_s = 1\} \subset S_\alpha$. (Because for any nonempty subset K of H, with $K \leqslant M$, the ordinal supremum of K belongs to H.)

Thus to complete the proof it suffices to show how to construct such a family $(S_\alpha)_{\alpha \in H}$. Let $V \in D$, $V = \prod_{s \in S} V_s$, such that $0 \in V_s, \forall s \in S$, and $G(S) \cap V \subset A$ or $G(S) \cap V \subset B$. We put $S_0 = \{s \in S \mid V_s \neq \{0,1\}\}$. Now let us suppose that for an ordinal $\lambda \in H$, $\lambda \neq 0$, we have the sets S_α for any $\alpha \ll \lambda$, satisfying the conditions:

a$_1$) $S_\alpha \leqslant 2^M, \forall \alpha \in H, \alpha \ll \lambda$ and $S_\beta \subset S_\alpha, \forall \alpha, \beta \in H, \beta \ll \alpha \ll \lambda$;

a$_2$) if $x \in G(S), x = (x_s)_{s \in S}$, and $\{s \in S \mid x_s = 1\} \subset S_\alpha$ for some $\alpha \in H$, with $\alpha \ll \alpha' \ll \lambda$, there is a set $V \in D$, $V = \prod_{s \in S} V_s$, such that $x \in V$, $\{s \in S \mid V_s \neq \{0,1\}\} \subset S_{\alpha'}$ and $G(S) \cap V \subset A$ or $G(S) \cap V \subset B$.

Let us construct S_λ. For each $x \in G(S), x = (x_s)_{s \in S}$, with $\{s \in S \mid x_s = 1\}$

$\subset T$, where $T=\cup\{S_\alpha \mid \alpha\in H, \alpha\ll\lambda\}$, we fix a set $V_x\in D, V_x=\prod_{s\in S} V_s^x$, such that $x\in V_x$ and $G(S)\cap V_x\subset A$ or $G(S)\cap V_x\subset B$. We put $S_\lambda=TU\bigcup_{x\in U}\{s\in S\mid V_s^x\neq\{0,1\}\}$, where $U=\{(x_s)_{s\in S}\in G(S)\mid\{s\in S\mid x_s=1\}\subset T\}$. Since $T\leqslant 2^M$ we have that $S_\lambda\leqslant 2^M$.

Thus, by applying the argument above and transfinite induction, there is a family $(S_\alpha)_{\alpha\in H}$ verifying the conditions a) and b).

5. From the precedent sections, we have that (I) implies (II) and (II) implies the GCH (Theorem 3). On the other hand, if the GCH (and, thus, the Axiom of Choice) is verified, by virtue of Theorem 4, we have that "for any infinite sets M, Y and S, with $M<Y\leqslant 2^M\leqslant S$, $G(S, Y)$ is a topological group and has property K" (because, necessarily, $Y\approx 2^M$). Thus, we proved the following theorem:

Theorem 5. *In the Zermelo-Fraenkel set theory, without the Axiom of Choice, the following propositions are equivalent*:

1) *proposition* (I);
2) *proposition* (II);
3) *Generalized Continuum Hypothesis.*

Remark. We thank Professor Newton C. A. da Costa for drawing our attention to the fact that the Axiom of Regularity is not used in the above proofs.

References

[1] O. T. Alas: Topological groups and uniform continuity (submitted to Portugaliae Mathematica).
[2] Paul J. Cohen: Set theory and the Continuum Hypothesis. W. A. Benjamin, Inc., New York (1966).
[3] W. W. Comfort and Kenneth A. Ross: Pseudocompactness and uniform continuity in topological groups. Pacific J. Math., **16**, 483–496 (1966).
[4] E. Kamke: Théorie des ensembles. Dunod, Paris (1964).