199. *Certain Convexoid Operators*

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1. Introduction. We call a bounded linear operator T on a complex Hilbert space H, according to [2], *paranormal* if

$$\|T^2x\| \geqq \|Tx\|^2 \tag{1}$$

for every unit vector x in H.

It is easy to verify that any hyponormal* operator is paranormal. In fact if T is hyponormal

$$\|Tx\|^2 = (T^*Tx, x) \leqq \|T^*Tx\| \leqq \|T^2x\|$$

for every unit vector x.

It is known that there exists a paranormal but non-hyponormal operator and every power of paranormal operator is again paranormal [2], also paranormal operator is normaloid*) [2] [9] and moreover paranormal operator is compact if some of its powers is compact [5] and that compact paranormal operator is normal [9], and the inverse of a paranormal is also [2] [9].

In [1] T. Ando has given an elegant algebraic characterization of paranormal operator and he has proved several interesting results. Some of them are as follows; a bounded linear operator T is normal if and only if both T and T^* are paranormal and they have the common kernel, and moreover a paranormal operator is normal if some of its power is normal as a generalization of Stampfli's result [12] in the case of hyponormal operator.

Following Halmos [7] the numerical range $W(T)$ is defined as follows:

$$W(T) = \{(Tx, x); \|x\| = 1\}.$$

An operator T is said to be *convexoid* [7] if

$$\overline{W(T)} = co\, \sigma(T)$$

where $co\, \sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of T and the $\overline{W(T)}$ means the closure of the set $W(T)$. An operator T is said to be *spectraloid* [7] if

$$w(T) = r(T)$$

or equivalently

$$w(T^n) = (w(T))^n \qquad (n = 1, 2, \cdots) \text{ [4]}$$

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*) An operator T is said to be *hyponormal* if $\|Tx\| \geqq \|T^*x\|$ for every vector x and *normaloid* if $\|T^n\| = \|T\|^n$ $(n = 1, 2, \cdots)$ [7].

where $w(T)$ and $r(T)$ mean the numerical radius and the spectral radius of T respectively as follows:

$$w(T)=sup\ \{|\lambda|\,;\ \lambda\in W(T)\}$$
$$r(T)=sup\ \{|\lambda|\,;\ \lambda\in\sigma(T)\}.$$

It is known that there exist convexoid operators which are not normaloid and vice versa and the classes of normaloids and convexoids are both contained in the class of spectraloids [7] and every hyponormal operator is convexoid [11] [13] [14].

In [2] there is given an example of *paranormal, non-hyponormal* and *convexoid* operator.

In this paper we shall give the following results:

(i) an alternative proof of the following characterization of convexoid operators: T is convexoid if and only if $T-\lambda$ is spectraloid for every complex λ (Theorem 1 [6])

(ii) an example of paranormal by T. Ando is also convexoid and related results.

2. In [6] we have given a characterization of convexoid operators and we shall give an alternative proof of this characterization in this section. In [6] the following theorems have been proved in collaboration with my colleague Nakamoto and we cite them here with his agreement.

Theorem 1 [6]. *An operator T is convexoid if and only if $T-\lambda$ is spectraloid for every complex λ.*

To prove the theorem stated above we need the following obvious lemma.

Lemma. *If X is any bounded closed set in the complex plane, then*

$$co\ X=\bigcap_{\mu}\{\lambda\,;\,|\lambda-\mu|\leqq\sup_{x\in X}|x-\mu|\}\tag{2}$$

where $co\ X$ means the convex hull of the set X.

Putting $X=\overline{W(T)}$ and $\sigma(T)$ in lemma respectively, we have following (3) and (4) without appealing to Williams's result [15] since $\overline{W(T)}$ is convex [7].

Theorem 2 [6]. *If T is an operator on a Hilbert space, then*

$$\overline{W(T)}=\bigcap_{\mu}\{\lambda\,;\,|\lambda-\mu|\leqq w(T-\mu)\}\tag{3}$$

$$co\ \sigma(T)=\bigcap_{\mu}\{\lambda\,;\,|\lambda-\mu|\leqq r(T-\mu)\}.\tag{4}$$

As we described in [6], theorem 1 is an immediate consequence of (3) and (4) since $co\ \sigma(T)-\lambda=co\ \sigma(T-\lambda)$ and every convexoid is spectraloid.

Corollary 1. (i) *An operator T is convexoid if $T-\lambda I$ is normaloid for every complex number λ* [8] [14].

(ii) *Every hyponormal operator is convexoid* [11] [13] [14].

Putting $\lambda=0$ in (3) and (4) we have the following corollary.

Corollary 2. (i) $0\in\overline{W(T)}$ *if and only if* $|\mu|\leqq w(T-\mu)$ *for all complex* μ [6].

(ii) $0\in co\,\sigma(T)$ *if and only if* $|\mu|\leqq r(T-\mu)$ *for all complex* μ.

Different characterizations of convexoid operators are known by Orland [10], Furuta [3] and we shall explain the correlation between three characterizations by simple and unified lemmas [3].

3. *Is every paranormal operator convexoid?* This question is given in [2] and the answer is not certain for the present author, in this section we attempt to make progress in solving this question.

1) *An example (T. Ando) of non-hyponormal, paranormal convexoid operator* [1]. T. Ando [1] has given the following concrete example as follows: when H is a complex Hilbert space, K denotes the infinite direct sum of copies of H, i.e. $K=\bigoplus_{k=1}^{\infty}H_k$ $(H_k\cong H)$.

Given two bounded positive operators A and B on H, the infinite matrix $T_{A,B,n}$ is defined on K, which assigns to a vector

$$x=(x_1,x_2,\cdots)\ \text{the vector}\ y=(y_1,y_2,\cdots)$$

such that $y_1=0$, $y_j=Ax_{j-1}$ $(1<j\leq n)$ and $y_j=Bx_{j-1}$ $(j>n)$, that is,

$$(5)\qquad T_{A,B,n}=\begin{pmatrix} 0 & & & & & & & & & \\ A & 0 & & & & & & & & \\ & A & \cdot & & & & & & & \\ & & A & \cdot & & & & & & \\ & & & \cdot & \cdot & & & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & & A & 0 & & & \\ & & & & & & B & 0 & & \\ & & & & & & & B & \cdot & \\ & & & & & & & & \cdot & \cdot \\ & & & & & & & & & \cdot\ \ \cdot \\ & & & & & & & & & \ \ \cdot\ \ \cdot \end{pmatrix}.$$

T. Ando shows that this operator $T_{A,B,n}$ is paranormal if and only if

$$(6)\qquad AB^2A-2\lambda A^2+\lambda^2\geqq 0\qquad(\lambda>0)$$

and that it is hyponormal if and only if $B^2\geqq A^2$. He observed the operator

$$T=T_{A,B,n}\ \text{with}\ A=C^{1/2},\quad B=(C^{-1/2}DC^{-1/2})^{1/2}$$

where

$$C=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\quad\text{and}\quad D=\begin{pmatrix}1 & 2\\ 2 & 8\end{pmatrix},$$

then this T is paranormal by (6), but the tensor product $T\otimes T$ is not paranormal.

At first we show that this paranormal operator T is convexoid and non-paranormal $T\otimes T$ is also convexoid as follows. Let μ be the proper value of positive operator B, sucn that, $B\varphi=\mu\varphi$, $\|B\|=\mu$. Take an arbitrary complex number λ such that $\|A\|<|\lambda|<\|B\|=\mu$ and we consider the vector Φ on K

$$\Phi=\left\{\frac{1}{\lambda^n}A^n\varphi,\ \frac{1}{\lambda^{n-1}}A^{n-1}\varphi,\ \cdots,\ \frac{1}{\lambda}A\varphi,\ \varphi,\ \frac{\lambda}{\mu}\varphi,\ \frac{\lambda^2}{\mu^2}\varphi,\ \cdots,\ \frac{\lambda^k}{\mu^k}\varphi,\ \cdots\right\}.$$

Then clearly Φ is a vector in $K=\bigoplus_{k=1}^{\infty}H_k$ and an easy calculation implies

$$\text{that } T^*\Phi=\lambda\Phi, \text{ so that } \lambda^*\in P_\sigma(T)\cup R_\sigma(T)\subset\sigma(T).$$

This assures that every complex number λ^* such that $\|A\|<|\lambda|<\|B\|=\mu$ is in the spectrum $\sigma(T)$ and so that the convex hull of the spectrum of T coincides with the disc $\{z\,;|z|\leq\mu\}$. On the other hand $\|T\|=\|B\|=\mu$, the closed numerical range of T, $\overline{W(T)}$ is contained in this disc. Hence T is convexoid.

Next we consider

$$co\ \sigma(T\otimes T)=co\ (\sigma(T)\cdot\sigma(T))=\{z:|z|\leqq\mu^2\},$$

on the other hand

$$\mu^2=\|T\|^2=\|T\otimes T\|\geqq|W(T\otimes T)|,$$

so that

$$\overline{W(T\otimes T)}=co\ \sigma(T\otimes T),$$

thus $T\otimes T$ is also convexoid.

2) *An another example of paranormal and convexoid operator.* Let T be the following operator T on $K=\bigoplus_{k=-\infty}^{\infty}H_k$ ($H_k\cong H$, where H is arbitrary dimensional Hilbert space and $\boxed{0}$ shows the place of the $(0,0)$ matrix element).

$$(7)\qquad T=\begin{pmatrix}\ddots & & & & & & & \\ & A & & & & & & \\ & & A & \boxed{0} & & & & \\ & & & A & & & & \\ & & & & B & & & \\ & & & & & B & & \\ & & & & & & B & \\ & & & & & & & \ddots\end{pmatrix}$$

where A and B are both positive operators on H such that satisfy (6) and $\|A\|>\|B\|$, then T is *paranormal* and *convexoid*. It is easy to verify that $\|T\|=\|A\|=\mu$, where μ is the approximate proper value of positive operator A such that

$$\|\varphi_n\|=1,\qquad \|A\varphi_n-\mu\varphi_n\|\to 0.$$

Take an arbitrary complex number λ such that $\|B\|<|\lambda|<\|A\|$. We consider the vector ψ_n in K as follows:

$$\psi_n=\left\{\cdots,\ \frac{\lambda^3}{\mu^3}\varphi_n,\ \frac{\lambda^2}{\mu^2}\varphi_n,\ \boxed{\frac{\lambda}{\mu}\varphi_n},\ \varphi_n,\ \frac{1}{\lambda}B\varphi_n,\ \frac{1}{\lambda^2}B^2\varphi_n,\ \frac{1}{\lambda^3}B^3\varphi_n,\ \cdots\right\}$$

where each component is a vector in H_k respectively and $\square$ means the place of the 0-th coordinate. By the simple calculation we have

$$\|T\psi_n-\lambda\psi_n\|^2=\|A\varphi_n-\mu\varphi_n\|^2\cdot\sum_{k=1}^{\infty}\left|\frac{\lambda}{\mu}\right|^{2k}\to 0.$$

We put $\Phi_n=\dfrac{\psi_n}{\|\psi_n\|}$. We can conclude $\|\Phi_n\|=1$ and

$$\|T\Phi_n-\lambda\Phi_n\|\to 0,$$

so that λ is an approximate proper value of T. This assures that every complex number λ such that $\|B\|<|\lambda|<\|A\|$ is in the spectrum $\sigma(T)$ and the convex hull of the spectrum of T coincides with the disc $\{z\,;|z|\leqq\mu\}$. On the other hand $\|T\|=\mu$, the closed numerical range of T, $\overline{W(T)}$ is contained in this disc. Hence T is convexoid and $\overline{T\otimes T}$ is also by the same argument in the preceding 1) and paranormality of T follows from (6).

Remark. (i) In (7) if A and B are both positive operators on H such that satisfy (6) and $\|A\|<\|B\|$, then T is also convexoid. In this case we have only to consider T^* instead of T since B is positive so that we can use the same method in 2).
(ii) The convexity of T in (7) remains valid for more wider class than that of positive operators A and B since the positivity of A and B is not essential in the above argument, in fact we may take an arbitrary operator for the one having less norm than the other, but we consider only paranormal operators induced by the positive operators A and B which satisfy the simple relation (6) so that we omit to describe it in detail.

4. T. Ando [1] has proved the following theorem.

Theorem. *If a paranormal operator T is doubly commuting with a hyponormal operator S, then TS is paranormal.*

Motivated by this theorem we may naturally come to mind the following question.

"*Can we replace hyponormality and paranormality by normaloid?*"
In this section we shall give a negative answer to this question.

Theorem 3. *There exist doubly commuting normaloid operators whose product is nilpotent.*

Proof. We put T and S as follows.

$$T=\begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & M\end{pmatrix},\qquad S=\begin{pmatrix} 0 & I & 0\\ 0 & 0 & 0\\ 0 & 0 & I\end{pmatrix}$$

where I, O and M are two by two matrices on the two dimensional space respectively as follows:

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad M=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad O=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

By simple calculations we can show that T and S doubly commute and the product TS is nilpotant as follows

$$TS=\begin{pmatrix}0 & I & 0\\ 0 & 0 & 0\\ 0 & 0 & M\end{pmatrix}.$$

As an immediate consequence of Theorem 3 we have the following Corollary.

Corollary 3. *Product TS is not always normaloid even if T and S are doubly commuting normaloids.*

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