195. *The Theory of Nuclear Spaces Treated by the Method of Ranked Space. III*

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In the papers [3] and [4], we defined the neighbourhood having a rank in the nuclear space Φ, and then we made the Φ a linear ranked space.

§ 4. The completion of the linear ranked space Φ, (1).

Definition 4. We say that a sequence $\{g_n\}$ in Φ is R-convergence having a limiting point zero, if there exists a fundamental sequence of neighbourhoods $\{V_{r_i}(0, r_i, m_i)\}$ such that $g_i \in V_{r_i}(0, r_i, m_i)$ for all i. And we denote it by $g_n \xrightarrow{R} 0$.

In the paper [4], we defined the equivalence of two R-Cauchy sequences in Φ, so that the set of all R-Cauchy sequences in Φ is divided into equivalence classes. We denote by $\hat{\Phi}$ the set of all these equivalence classes.

Now, suppose $\hat{g}, \hat{f} \in \hat{\Phi}$ and let $\{g_n\}$ and $\{f_n\}$ be two R-Cauchy sequences in Φ which are in the equivalence classes $\hat{g}$ and $\hat{f}$, respectively. Then $\{g_n + f_n\}$ is an R-Cauchy sequence. Moreover if $\{g'_n\}$ and $\{f'_n\}$ are R-Cauchy sequences equivalent to $\{g_n\}$ and $\{f_n\}$ respectively, then $\{g'_n + f'_n\}$ is equivalent to $\{g_n + f_n\}$. Thus we can define $\hat{g} + \hat{f}$ as the equivalence class which contains $\{g_n + f_n\}$, and the definition depends only on $\hat{g}, \hat{f}$, not on the particular choice of $\{g_n\}, \{f_n\}$. Likewise, for any scalar λ, we define $\lambda\hat{g}$ as the equivalence class which contains $\{\lambda g_n\}$. The zero element of $\hat{\Phi}$ is the unique equivalence class all of whose members $\{g_n\}$ are such that $g_n \xrightarrow{R} 0$.

Now, we shall define a neighbourhood with rank i in $\hat{\Phi}$.

Definition 5. We define a neighbourhood, $\hat{V}_i(0, r, m)$, of the origin in $\hat{\Phi}$. $\hat{g} \in \hat{V}_i(0, r, m)$ means that for an R-Cauchy sequence $\{g_n\}$ belonging to $\hat{g}$, there exist some number r', $0 < r' < r$ and some integer N such that the relation $n \geqq N$ implies $g_n \in V_i(0, r', m)$. And we call $\hat{V}_i(0, r, m)$ a neighbourhood with rank i of the origin in $\hat{\Phi}$.

Moreover we define that the neighbourhood with rank 0, which is denoted by $\hat{V}_0$, is always the space $\hat{\Phi}$.

We shall show that in the definition above, every R-Cauchy sequence $\{g_n\}$ belonging to $\hat{g}$ has some number r', $0 < r' < r$ and some integer N such that the relation $n \geqq N$ implies $g_n \in V_i(0, r', m)$.

Suppose that two R-Cauchy sequences $\{g_n\}$ and $\{f_n\}$ belong to the equivalence class $\hat{g}$ and there exist some number r', $0<r'<r$ and some integer N such that the relation $n \geqq N$ implies $g_n \in V_i(0, r', m)$.

By Definition 3 in [4], there exists a fundamental sequence $\{V_{r_i}(0, r_i, m_i)\}$ such that $g_i - f_i \in V_{r_i}(0, r_i, m_i)$ for all integer i. By Lemma 10 in [4], there exists some $V_{r_j}(0, r_j, m_j)$ in $\{V_{r_i}(0, r_i, m_i)\}$ such that $V_i(0, r', m) \supseteqq V_{r_j}(0, r_j, m_j)$ with $m<m_j$, $i<\gamma_j$ and $r'>r_j$. And then we have $V_i(0, r', m) \supseteqq V_{r_n}(0, r_n, m_n)$ for $n \geqq j$.

Since $r_i \downarrow 0$, there exists some integer l such that $r-r'>r_l$, $j<l$ and $N<l$. And hence for all $n>l$, we have

$$f_n = (f_n - g_n) + g_n \in V_{r_n}(0, r_n, m_n) + V_i(0, r', m)$$
$$\subset V_i(0, r_n, m_n) + V_i(0, r', m) \subset V_i(0, r_n + r', \min(m_n, m)) \subset V_i(0, r_n + r', m).$$

Since $r_n + r' < r$, we assert.

Now, we define $\hat{V}_i(\hat{g}, r, m) = \hat{g} + \hat{V}_i(0, r, m)$ as a neighbourhood of $\hat{g}$ in $\hat{\Phi}$.

Lemma 14. *We have* $r\hat{V}_i(0, 1, m) = \hat{V}_i(0, r, m)$ *for any* $r>0$ *and any neighbourhood* $\hat{V}_i(0, r, m)$.

Proof. Let $\hat{g}$ belong to $\hat{V}_i(0, r, m)$. Then for an R-Cauchy sequence $\{g_n\}$ belonging to $\hat{g}$, there exist some number r_0, $0<r_0<r$ and some integer N such that the relation $n \geqq N$ implies $g_n \in V_i(0, r_0, m)$. And then we have

$$g_n/r \in V_i(0, r_0/r, m) \qquad \text{for } n \geqq N.$$

Hence it is clear that we have $\hat{g}/r \in \hat{V}_i(0, 1, m)$ by Definition 5. Consequently we obtain $\hat{V}_i(0, r, m) \subseteqq r\hat{V}_i(0, 1, m)$.

Conversely, let $\hat{g}$ belong to $r\hat{V}_i(0, 1, m)$. Then we have $\hat{g}/r \in \hat{V}_i(0, 1, m)$. Hence for an R-Cauchy sequence $\{g_n\}$ belonging to $\hat{g}$, there exist some number r_0, $0<r_0<1$, and some integer N such that the relation $n \geqq N$ implies $g_n/r \in V_i(0, r_0, m)$.

And then we have $g_n \in V_i(0, rr_0, m)$ for $n \geqq N$. Hence it shows that $\hat{g} \in \hat{V}_i(0, r, m)$ since $0<rr_0<r$. Consequently we obtain

$$r\hat{V}_i(0, 1, m) \subseteqq \hat{V}_i(0, r, m).$$

Lemma 15. *We have* $\hat{V}_j(0, 1, m) \supseteqq \hat{V}_i(0, 1, m)$ *if* $j \leqq i$.

Lemma 16. *We have* $\hat{V}_i(0, 1, m') \supseteqq \hat{V}_i(0, 1, m)$ *if* $m' \leqq m$.

Lemma 17. *We have* $\hat{V}_i(0, r, m) \supseteqq \hat{V}_i(0, r', m)$ *if* $r' \leqq r$.

Now, we shall define the fundamental sequence of neighbourhoods in $\hat{\Phi}$.

Definition 6. When a sequence of neighbourhoods $\{\hat{V}_{r_i}(0, r_i, m_i)\}$ in $\hat{\Phi}$ satisfies the following conditions, it is called the fundamental sequence in $\hat{\Phi}$,

(1) there exists some integer i_0 such that
$\hat{V}_{r_i}(0, r_i, m_i) = \hat{V}_0$ for $0 \leqq i \leqq i_0$,

(2) $\gamma_i \leqq \gamma_{i+1}$ for $i > i_0$ and $\gamma_i \to \infty$,

(3) $r_i \geqq r_{i+1}$ for $i > i_0$ and $r_i \to 0$,

(4) $m_i \leqq m_{i+1}$ for $i > i_0$ and $m_i \to \infty$.

Lemma 18. *If $\{\hat{V}_{\gamma_i}(0, r_i, m_i)\}$ is a fundamental sequence of neighbourhood in $\hat{\Phi}$, then $\hat{g} \in \hat{V}_{\gamma i}(0, r_i, m_i)$ for every i implies $\hat{g} = 0$.*

Proof. By Definition 6, there exists some integer i_0 such that $\hat{V}_{\gamma_i}(0, r_i, m_i) \neq \hat{V}_0$ for $i > i_0$. Since $\hat{g}$ belongs to $\hat{V}_{\gamma_i}(0, r_i, m_i)$, for an R-Cauchy sequence $\{g_n\}$ belonging to $\hat{g}$ there exist some number l_i, $0 < l_i < r_i$ and some integer N_i with $i_0 < N_i < N_{i+1}$ such that the relation $n \geqq N_i$ implies $g_n \in V_{\gamma_i}(0, l_i, m_i)$.

If we set $V_{\gamma'_i}(0, r'_i, m'_i) = V_0$ for $0 \leqq i < N_1$,

$$V_{\gamma'_i}(0, r'_i, m'_i) = V_{\gamma_1}(0, r_1, m_1) \qquad \text{for } N_1 \leqq i < N_2$$

and

$$V_{\gamma'_i}(0, r'_i, m'_i) = V_{\gamma_2}(0, r_2, m_2) \qquad \text{for } N_2 \leqq i < N_3, \text{ etc.}$$

Thus since $\{V_{\gamma'_i}(0, r'_i, m'_i)\}$ is the fundamental, the R-Canchy sequence $\{g_n\}$ belongs to the zero element in $\hat{\Phi}$.

Lemma 19. (1) *$\hat{V}_i(0, r, m)$ is circled.*

(2) *$\hat{V}_i(0, r, m) + \hat{V}_i(0, r', m) \subseteqq \hat{V}_i(0, r + r', \min(m, m'))$.*

Proof. (1) Let $\hat{g}$ belong to $\hat{V}_i(0, r, m)$. Then by Definition 5, there exist some number r_0, $0 < r_0 < r$ and some integer N such that the relation $n \geqq N$ implies $g_n \in V_i(0, r_0, m)$. Since $V_i(0, r_0, m)$ is circled, the relations $n \geqq N$ and $|\alpha| \leqq 1$ imply that αg_n belongs to $V_i(0, r_0, m)$. Consequently, $\alpha \hat{g}$ belongs to $\hat{V}_i(0, r, m)$.

(2) The relations $\hat{g} \in \hat{V}_i(0, r, m)$ and $\hat{f} \in \hat{V}_i(0, r', m')$ imply that for two R-Cauchy sequences $\{g_n\} \in \hat{g}$ and $\{f_n\} \in \hat{f}$, there exist some numbers r_0 and r'_0 with $0 < r_0 < r$ and $0 < r_0' < r'$ respectively, and some integer N such that the relation $n \geqq N$ implies $g_n \in V_i(0, r_0, m)$ and $f_n \in V_i(0, r'_0, m')$. And hence we have, for $n \geqq N$, $g_n + f_n \in V_i(0, r_0, m) + V_i(0, r'_0, m') \subset V_i(0, r_0 + r'_0, \min(m, m'))$. Since the R-Cauchy sequence $\{g_n + f_n\}$ belongs to $\hat{g} + \hat{f}$, we have $\hat{g} + \hat{f} \in \hat{V}_i(0, r + r', \min(m, m'))$.

We can show in the same manner in [4], that the ranked space $\hat{\Phi}$ is the linear ranked space.

Theorem 1. *The linear ranked space $\hat{\Phi}$ is complete, that is, any R-Cauchy sequence of elements in $\hat{\Phi}$ has a limiting element in $\hat{\Phi}$.*

Proof. Let $\{\hat{g}_k\}$ be the R-Cauchy sequence in $\hat{\Phi}$. Then by Definition 6, there exists a fundamental sequence $\{\hat{V}_{\gamma_i}(0, r_i, m_i)\}$, where $\hat{V}_{\gamma_i}(0, r_i, m_i) = \hat{V}_0$ for $0 \leqq i \leqq i_0$ and $\hat{V}_{\gamma_i}(0, r_i, m_i) \neq \hat{V}_0$ for $i > i_0$, such that the relations $k \geqq i$ and $h \geqq i$ imply $\hat{g}_k - \hat{g}_h \in \hat{V}_{\gamma_i}(0, r_i, m_i)$. If two R-Cauchy sequences $\{g_n^{(k)}\}_n$ and $\{g_n^{(h)}\}_n$ belong to $\hat{g}_k$ and $\hat{g}_h$ respectively, the R-Cauchy sequence $\{g_n^{(k)} - g_n^{(h)}\}_n$ belongs to $(\hat{g}_k - \hat{g}_h)$. Hence by Definition 5, there exist some number r, $0 < r < r_i$ and some integer N such that the relation $n \geqq N$ implies $g_n^{(k)} - g_n^{(h)} \in V_{\gamma_i}(0, r, m_i)$.

On the other hand, since $\{g_n^{(k)}\}_n$ is an R-Cauchy sequence in Φ, there exists a fundamental sequence of neighbourhoods, $\{V_{\gamma_i(k)}(0, r_i^{(k)}, m_i^{(k)})\}_i$ in Φ such that the relations $n \geqq i$ and $m \geqq i$ imply

$$g_n^{(k)} - g_m^{(k)} \in V_{\gamma_i(k)}(0, r_i^{(k)}, m_i^{(k)}).$$

And for all integer k we can take $V_{\gamma'_k}(0, r'_k, m'_k)$ in $\{V_{\gamma_i(k)}(0, r_i^{(k)}, m_i^{(k)})\}$ such that $\gamma'_k > k$, $r'_k < 1/k$ and $m'_k > k$, where $\gamma'_k < \gamma'_{k+1}$, $r'_k > r'_{k+1}$ and $m'_k < m'_{k+1}$.

And then we make $g_{n(k)}^{(k)}$ satisfy that the relation $n \geqq n(k)$ implies $g_{n(k)}^{(k)} - g_n^{(k)} \in V_{\gamma'_k}(0, r'_k, m'_k)$.

If we set $f_k = g_{n(k)}^{(k)}$, $\{f_k\}$ is an R-Cauchy sequence in Φ. Because, let $V_i(0, R, m) \neq V_0$ be a arbitrary neighbourhood of the origin in Φ. Since $\{V_{\gamma_i}(0, r_i, m_i)\}$ is the fundamental sequence of neighbourhoods, we can find $V_{\gamma_j}(0, r_j, m_j)$ such that $V_i(0, R, m) \supseteqq V_{\gamma_j}(0, r_j, m_j)$ and $R > r_j$.

Since the relations $k \geqq j$ and $h \geqq j$ imply $\hat{g}_k - \hat{g}_h \in \hat{V}_{\gamma_j}(0, r_j, m_j)$, there exist some number $r = r(k, h)$, $0 < r < r_j$ and some integer $N = N(k, h)$ such that the relation $n \geqq N$ implies $g_n^{(k)} - g_n^{(h)} \in V_{\gamma_j}(0, r, m_j)$.

And then we have, for $n \geqq \max(N(k, h), n(k), n(h))$

$$f_k - f_h = g_{n(k)}^{(k)} - g_{n(h)}^{(h)} = (g_{n(k)}^{(k)} - g_n^{(k)}) + (g_n^{(k)} - g_n^{(h)}) + (g_n^{(h)} - g_{n(h)}^{(h)})$$
$$\in V_{\gamma'_k}(0, r'_k, m'_k) + V_{\gamma_j}(0, r, m_j) + V_{\gamma'_h}(0, r'_h, m'_h).$$

Since we have $\gamma'_k > k$, $r'_k < 1/k$ and $m'_k > k$ for any integer k, we can take k and h such that $\gamma'_k > \gamma_j$, $\gamma'_h > \gamma_j$, $m'_k > m_j$, $m'_h > m_j$ and $r'_k + r'_h + r < 1/k + 1/h + r_j < R$. And hence we obtain

$$V_{\gamma'_k}(0, r'_k, m'_k) + V_{\gamma_j}(0, r, m_j) + V_{\gamma'_h}(0, r'_h, m'_h)$$
$$\subset V_{\gamma_j}(0, r'_k + r + r'_h, m_j) \subset V_{\gamma_j}(0, R, m_j) \subset V_i(0, R, m).$$

Consequently $\{f_k\}$ is an R-Cauchy sequence by Lemma 13 in [4].

Now, we shall prove that if the R-Cauchy sequence $\{f_k\}$ belongs to $\hat{f}$ in $\hat{\Phi}$, $\hat{f}$ is a limiting element of the R-Cauchy sequence $\{\hat{g}_k\}$.

We have known the following facts in the proof above.

(1) For any $V_{\gamma_i}(0, r_i, m_i)$ in $\{V_{\gamma_i}(0, r_i, m_i)\}$ there exist some number r, $0 < r < r_i$ and some integer N such that the relations $m \geqq N$, $l \geqq i$ and $k \geqq i$ imply $g_m^{(l)} - g_m^{(k)} \in V_{\gamma_i}(0, r, m_i)$.

(2) The relations $m \geqq j$ and $l \geqq j$ imply

$$g_m^{(k)} - g_l^{(k)} \in V_{\gamma_j(k)}(0, r_j^{(k)}, m_j^{(k)}).$$

(3) The relation $m \geqq n(l)$ implies

$$g_{n(l)}^{(l)} - g_m^{(l)} \in V_{\gamma'_l}(0, r'_l, m'_l).$$

And then for any $V_{\gamma_i}(0, r_i, m_i)$ and any integer $k \geqq i$, we take j and l such that $1/l + r + r_j^{(k)} < r_i$, $m_j^{(k)} > m_i$, $m'_l > m_i$, $\gamma_j(k) > \gamma_i$ and $\gamma'_l > \gamma_i$.

Moreover we can take l and m such that $l > \max(j, i)$ and $m > \max(n(l), N, j)$.

Hence we have for all $l \geqq \max(j, i)$ and $k \geqq i$,

$$f_l - g_l^{(k)} = g_{n(l)}^{(l)} - g_l^{(k)} = (g_{n(l)}^{(l)} - g_m^{(l)}) + (g_m^{(l)} - g_m^{(k)}) + (g_m^{(k)} - g_l^{(k)})$$
$$\in V_{\gamma'_l}(0, r'_l, m'_l) + V_{\gamma_i}(0, r, m_i) + V_{\gamma_j(k)}(0, r_j^{(k)}, m_j^{(k)})$$

$$\subset V_{r_i}(0, r'_l, m_i) + V_{r_i}(0, r, m_i) + V_{r_i}(0, r_j^{(k)}, m_i)$$
$$\subset V_{r_i}(0, r'_l + r + r_j^{(k)}, m_i) \subset V_{r_i}(0, 1/l + r + r_j^{(k)}, m_i).$$

And then the relation $k \geqq i$ implies $\hat{f} - \hat{g}_k \in \hat{V}_{r_i}(0, r_i, m_i)$. Consequently the proof is complete.

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