187. On the Bi-ideals in Semigroups. II

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This note is a continuation of a recent paper of the author [8]. In this series some important results about bi-ideals of semigroups are summarized and some new results are announced. We adopt the standard notation and terminology due to A. H. Clifford and G. B. Preston [3].

Theorem 1. *Let S be a semigroup. Suppose that B is a bi-ideal, T is a subsemigroup of S, and the intersection $A=B\cap T$ is not empty. Then A is a bi-ideal of T.*

This is a consequence of a theorem concerning (m, n)-ideals (cf. the author [7], Theorem 1).

The following result shows that the existence of proper bi-ideal (in some cases) implies that of proper left (and right) ideal.

Theorem 2. *Suppose that A is a proper bi-ideal of a semigroup S, not being a left (right) ideal of S. Then the product BS (SB) is a proper right (left) ideal of S.*

The author proved the following statement [6].

Theorem 3. *Let S be a regular semigroup. Then every bi-ideal of S is a quasi-ideal, and conversely.*

K. M. Kapp [5] proved the following two results.

Theorem 4. *If S is a left simple semigroup, then every bi-ideal B of S is a right ideal.*

Theorem 5. *Let S be a semigroup with zero. If S is left 0-simple, then the sets of bi-ideals and quasi-ideals of S coincide.*

The following example shows that there exists such a bi-ideal which is not quasi-ideal.

Example 1. Let S be the semigroup of four elements 0, 1, 2, 3 with multiplication table

	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

.

It is easy to see that the subsemigroup $B=\{0, 2\}$ is a bi-ideal of S

which is not a quasi-ideal of S. Actually B is a two-sided ideal of the two-sided ideal $I=\{0,1,2\}$ of S, i.e. B is a 2-ideal of S (see the author [6]).

Utilizing Theorem 3, J. Dénes [4] proved the following result.

Theorem 6. *In the symmetric semigroup F_n of transformations of a set of n elements every bi-ideal is a left ideal.*

A generalization of this assertion reads as follows.

Theorem 7. *Let S be a regular right duo semigroup. Then every bi-ideal B of S is a left ideal.*

A semigroup S is called *right duo* if every right ideal R of S is two-sided.

Example 2. The semigroup S of the four elements $0,1,2,3$ with the multiplication table

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

is a regular right duo semigroup. It is easy to see that S has the property:

(P) *Every non-empty subset of S is a subsemigroup.*

Thus, by a theorem of L. Rédei ([10], Theorem 50), S is a chain of right zero semigroups.

Theorem 8. *Let S be a regular right duo semigroup, and let A be a non-empty subset of S. Then A is an (m,n)-ideal of S if and only if it is a $(0,n)$-ideal of S.*

The following result is a consequence of Theorem 8.

Theorem 9. *A non-empty subset T of the symmetric semigroup F_n is a (p,q)-ideal of F_n if and only if T is a $(0,q)$-ideal of F_n (p,q are arbitrary non-negative integers).*

The next result is a criterion for a right duo semigroup to be a semilattice of groups.

Theorem 10. *A right duo semigroup S is a semilattice of groups if and only if the condition*

(1) $$B\cap I=BI$$

holds for every bi-ideal B and every two-sided ideal I of S.

The following two theorems characterize the class of semigroups that are semilattices of groups in terms of bi-ideals.

Theorem 11. *A semigroup S is a semilattice of groups if and only if the intersection of any two bi-ideals of S is equal to their product.*

Theorem 12. *A semigroup S is a semilattice of groups if and only if the set of bi-ideals of S is a semilattice under the multiplication of subsets.*

Next the class of regular semigroups will be characterized in terms of bi-ideals.

Theorem 13. *A semigroup S is regular if and only if the relation*

$$BSB=B \tag{2}$$

holds for each bi-ideal B of S.[1)]

The following two results are due to J. Calais [1] [2].

Theorem 14. *A semigroup S is regular if and only if each left ideal and each right ideal of S is globally idempotent, and the product RL is a quasi-ideal of S for every left ideal L and every right ideal R of S.*

Theorem 15. *For a semigroup S the sets of bi-ideals and quasi-ideals coincide if and only if $B(x, y)=Q(x, y)$ for every couple x, y in S.*

$B(x, y)$ and $Q(x, y)$ denote the smallest bi-ideal and quasi-ideal of S containing the elements x, y of S.

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1) This result remains true with quasi-ideal instead of bi-ideal (cf. Luh [9]).