226. A Note on Nuclear Operators in Hilbert Space

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1. Introduction. Let $\mathfrak{H}$ be a separable Hilbert space and T a (bounded linear) operator on $\mathfrak{H}$. Then T is said to be *nuclear* (or of *trace class*) if there is an orthonormal basis (e_n) in $\mathfrak{H}$ such that

$$(1) \qquad \sum_{n=1}^{\infty}(|T|e_n|e_n)<+\infty,$$

where $|T|$ is the absolute of T. Let $\mathfrak{T}$ be the set of all nuclear operators. Then $\mathfrak{T}$ is an algebraical ideal of $\mathfrak{B}(\mathfrak{H})$ which is the algebra of all operators on $\mathfrak{H}$. If $T\in\mathfrak{T}$, then there are orthonormal sets (e_n) and (f_n) such that

$$(2) \qquad T=\sum_{n=1}^{\infty}a_n e_n\otimes f_n,$$

where (a_n) is a sequence of positive numbers such as

$$(3) \qquad \sum_{n=1}^{\infty}a_n<+\infty$$

and

$$(4) \qquad (e\otimes f)g=(g|f)e,$$

in the notation of Schatten [2].

According to [1], K. Maurin conjectured that an operator T in nuclear if and only if T satisfies

$$(5) \qquad \sum_{n=1}^{\infty}\|Te_n\|<+\infty$$

for any orthonormal set (e_n). The conjecture is recently disproved by J. R. Holub [1]:

Theorem 1. *There is a nuclear operator T which does not satisfy* (5) *for an orthonormal basis (e_n).*

Moreover, Holub [1] proves by virtue of a result of J. Lindenstrauss and A. Pelczynski on absolutely summing operators.

Theorem 2. *An operator T is nuclear if and only if there is an orthonormal basis (e_n) which satisfies* (5).

In the present note, we shall give simplified proofs of the above theorems in §§ 2–3. Incidentally, we shall characterize operators which satisfy Maurin's conjecture in § 4.

2. Proof of Theorem 2. It seems to us that Theorem 2 is already known, e.g. [3, Ex. 3–B, No. 30, p. 69]. However, for the sake of

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completeness, we shall give here a proof without appealing to Lindenstrauss-Pelczynski's result. Let T be a nuclear operator expressed in (2). For (f_n), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\|Tf_n\| &= \sum_{n=1}^{\infty}\left\|\sum_{m=1}^{\infty}a_m(e_m\otimes f_m)f_n\right\| \\ &= \sum_{n=1}^{\infty}\left\|\sum_{m=1}^{\infty}a_m(f_n\,|\,f_m)e_m\right\| \\ &= \sum_{n=1}^{\infty}\|a_ne_n\| \\ &= \sum_{n=1}^{\infty}a_n < +\infty,\end{aligned}$$

so that (f_m) satisfies (5).

Conversely, suppose that T satisfies (5) for an orthonormal basis (h_m). Then we have

$$\begin{aligned}\sum_{n=1}^{\infty}(|T|\,h_n\,|\,h_n) &= \sum_{n=1}^{\infty}(VTh_n\,|\,h_n) \\ &\leqq \sum_{n=1}^{\infty}\|Th_n\| < +\infty,\end{aligned}$$

where V is a partial isometry such that $|T|=VT$. Hence T is nuclear.

3. Proof of Theorem 1. We shall show that a one-dimensional projection

$$P=e\otimes e \qquad (\|e\|=1) \tag{6}$$

disproves the conjecture. For an orthonormal basis (e_n), put

$$e=\frac{\sqrt{6}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e_n.$$

Then we have

$$\begin{aligned}\sum_{n=1}^{\infty}\|Pe\| &= \sum_{n=1}^{\infty}\|(e_n\,|\,e)e\| \\ &= \frac{\sqrt{6}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}=+\infty.\end{aligned}$$

4. The class of operators satisfying Maurin's conjecture. Let $\mathfrak{M}$ be the set of operators which satisfy (5) for every orthonormal set (e_n). We shall show

Theorem 3. *$\mathfrak{M}$ contains no non-trivial operator.*

At first, we shall show a lemma which is suggested by § 3.

Lemma 1. *$\mathfrak{M}$ excludes any one-dimensional projection.*

Proof. Let $P=e\otimes e$ with $\|e\|=1$. Take an orthonormal set (e_n) such that

$$(e\,|\,e_n)=\frac{\sqrt{6}}{n\pi}.$$

Then we have

$$\sum_{n=1}^{\infty} \|Pe_n\| = \sum_{n=1}^{\infty} \|(e \otimes e)e_n\|$$
$$= \sum_{n=1}^{\infty} |(e \mid e_n)|$$
$$= \sum_{n=1}^{\infty} \frac{\sqrt{6}}{n\pi} = +\infty.$$

Hence P is not in $\mathfrak{M}$.

Lemma 2. *$\mathfrak{M}$ is an algebraical left ideal of $\mathfrak{B}(\mathfrak{H})$.*

Proof. Clearly, $\mathfrak{M}$ is a linear space. If $A \in \mathfrak{B}(\mathfrak{H})$ and $B \in \mathfrak{M}$, then we have

$$\sum_{n=1}^{\infty} \|ABe_n\| \leqq \sum_{n=1}^{\infty} \|A\| \, \|Be_n\|$$
$$= \|A\| \sum_{n=1}^{\infty} \|Be_n\| < +\infty$$

for every orthonormal basis (e_n). Hence $AB \in \mathfrak{M}$.

Proof of Theorem 3. Let $T \in \mathfrak{M}$. Then T is compact. By Lemma 2, we can assume without loss of generality that T is positive. By [2], we can express T in

$$T = \sum_{n=1}^{\infty} a_n e_n \otimes e_n,$$

where (e_n) is orthonormal and $a_n \downarrow 0$. Put

$$Q = e_1 \otimes e_1.$$

Then we have

$$QT = a_1 e_1 \otimes e_1.$$

By Lemma 2, $QT \in \mathfrak{M}$. On the other hand, $QT \notin \mathfrak{M}$ by Lemma 1. This contradiction proves the theorem.

References

[1] J. R. Holub: Characterization of nuclear operators in Hilbert space. Rev. Roum. Math. Pure Appl., **16**, 687–690 (1971).

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[3] O. Takenouchi: Exercises in Functional Analysis (in Japanese). Asakura, Tokyo (1968).