214. Localization Principle for Differential Complexes and Its Application

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0. In this note we announce a general principle for proving the exactness or the partial exactness, in sheaf level, of complexes with first order differential operators as their differentiations, and its application to the Dolbeault type sequences for real submanifolds in complex manifolds.

Throughout this note we assume the differentiability of class C^∞ for manifolds, vector bundles, differential operators and so on, unless otherwise stated.   When F is a vector bundle over a manifold M, $C^\infty(F)$ denotes the set of smooth sections of F over M, while $C^\infty(U, F)$ denotes the set of smooth sections of F over an open subset U of M.

1. Localization principle. We shall first fix a differential complex

$$\cdots \longrightarrow E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \longrightarrow \cdots$$

where E^i, $i \in Z$ are (complex) vector bundles over a manifold M, and $\partial^i : E^i \to E^{i+1}$ are first order differential operators such that $\partial^{i+1} \cdot \partial^i = 0$. For brevity we denote this complex by $E\cdot$ and the corresponding complex for sections $(C^\infty(E^i))_{i \in Z}$ by $C^\infty(E\cdot)$.   Now we shall list up the definitions which are needed to formulate the fundamental theorems.

Definition 1. A subcomplex $A\cdot = (A^i)_{i \in Z}$ of $C^\infty(E\cdot)$ is called *complete* if and only if (1) A^i, $i \in Z$ are complete locally convex topological vector spaces such that the inclusions $A^i \subseteq C^\infty(E^i)$ are continuous and (2) $\partial^i | A^i : A^i \to A^{i+1}$ is continuous for these topologies.

Definition 2. Let $A\cdot$, $B\cdot$ be two subcomplexes of $C^\infty(E\cdot)$ such that $A\cdot \subset B\cdot$ and let s^i, $i \leq q$ be linear maps from A^i to B^{i-1}.   The family of maps $s = (s^i)_{i \leq q}$ is said to be a $(-\infty, q)$-*homotopy for the inclusion* $A\cdot \subseteq B\cdot$ if and only if (1) $(\partial^{i-1} s^i + s^{i+1} \partial^i)u = u$ for $i < q$ and for $u \in A^i$, and (2) $\partial^{q-1} s^q u = u$ for $u \in A^q$ such that $\partial^q u = 0$.   In case $A\cdot$, $B\cdot$ are both complete, s is said to be continuous when each map s^i, $i \leq q$ is continuous.

Definition 3. Let $A\cdot \subset B\cdot \subset C\cdot$ be subcomplexes of $C^\infty(E\cdot)$ such that $A\cdot \subset B\cdot \subset C\cdot$.   Let further s^i, $i > q$ be linear maps from C^i into C^{i-1} and s^q a linear maps from C^q into $C^\infty(E^{q-1})$.   The family of maps $s = (s^i)_{i \geq q}$ is called a (q, ∞)-*homotopy of* $C\cdot$ *modulo* $(A\cdot, B\cdot)$ if and only if (1) $s^i(A^i) \subset B^{i-1}$ for $i > q$ and (2) $(\partial^{i-1} s^i + s^{i+1} \partial^i)u \equiv u$ mod. $B\cdot$ for $i \geq q$ and for $u \in C^i$.   In case $C\cdot$ is a complete subcomplex and each s^i, $i \geq q$

is continuous, we say that s is continuous.

Note that the conditions are rather delicate at the extreme point $i=q$ in the last two definitions.

Now let F be a vector bundle over M and Ω an open subset in C. Then we have canonical identification $C^\infty(\Omega, C^\infty(F))=C^\infty(\Omega\times M, \tilde{F})$ where $\tilde{F}$ is the pull back of the bundle F by the projection $\pi: C\times M\to M$. Given a smooth function f on M, we obtain a map which assigns an element, denoted by $[V]_f$, of $C^\infty(F)$ to each $V\in C^\infty(C, C^\infty(F))$, by setting

$$[V]_f=\tilde{\pi}\cdot V\cdot(f\times 1_M)$$

where $\tilde{\pi}$ denotes the projection from $\tilde{F}$ onto F. For a locally convex space A, $C^\infty(\Omega, A)$ denotes the space of smooth A-valued funcitons on Ω with the topology of uniform convergence of all derivatives on compact subsets. If $A\subsetneq C^\infty(F)$ is continuous, then $C^\infty(\Omega, A)$ is a subspace of $C^\infty(\Omega\times M, \tilde{F})=C^\infty(\Omega, C^\infty(F))$ and the inclusion $C^\infty(\Omega, A)\subsetneq C^\infty(\Omega, C^\infty(F))$ is continuous. If no topology is specified on $A\subset C^\infty(F)$, we assume that A has the topology induced from $C^\infty(F)$.

Definition 4. Let $A\cdot$ be a subcomplex of $C^\infty(E\cdot)$. A function $f\in C^\infty(M)$ is $A\cdot$-*admissible* if and only if the following conditions are fulfilled for $i\in Z$:

(1) $fu, \tilde{f}u, \varphi(f)u\in A^i$ for any $u\in A^i$ and for $\varphi\in C_0^\infty(C)$.

(2) $\chi(\zeta)\varphi(f)u/(\zeta-f)\in C_0^\infty(C, A^i)$ for $u\in A^i$ and for $\varphi, \chi\in C_0^\infty(C)$ such that supp $\varphi\cap$ supp $\chi=\phi$.

(3) $[V]_f\in A^i$ for every $V\in C_0^\infty(C, A^i)$.

In case $A\cdot$ is a complete subcomplex of $C^\infty(E\cdot)$ we require the continuity of maps $u\mapsto fu,\ u\mapsto \tilde{f}u,\ u\mapsto\varphi(f)u,\ u\mapsto\chi(\zeta)\varphi(f)u/(\zeta-f),\ V\mapsto[V]_f$ in (1), (2) and (3).

Definition 5. A function $f\in C^\infty(U)$ on an open subset U of M is called $E\cdot$-*analytic* in U if and only if $\partial^i(fu)=f\partial^i u$ for any $i\in Z$ and $u\in C^\infty(U, E^i)$.

The $E\cdot$-analyticity is a local property and the product of two $E\cdot$-analytic functions is again $E\cdot$-analytic, in fact there is a way to define $E\cdot$-analytic functions to be solutions of a homogeneous first order differential equation.

To state the fundamental theorems we still need some notation: Let K be a closed subset of M, and $A\cdot$ a subcomplex of $C^\infty(E\cdot)$. Then $A\cdot\{K\}$ is a subcomplex of $A\cdot$ whose terms $A^i\{K\}$, $i\in Z$, are given by

$$A^i\{K\}=\{u\in A^i\,|\,\text{supp } u\subset K\}.$$

If $A\cdot$ is complete, then $A\cdot\{K\}$ is also complete. Let now $f=(f_1, \cdots, f_\rho)$ be a ρ-tuple of functions on M. Then $P(f)$ denotes the open set $\{z\in M\,|\,|f_1(z)|<1, \cdots, |f_\rho(z)|<1\}$ and $P[f]$ denotes the closed set $\{z\in M\,|\,|f_1(z)|\leq 1, \cdots, |f_\rho(z)|\leq 1\}$. Now our main theorems are stated as follows:

Theorem 1. *Let $f=(f_1, \cdots, f_\rho)$ be a ρ-tuple of $E\cdot$-analytic func-*

tions which are all $A\cdot$-admissible for a complete subcomplex $A\cdot$ of $C^\infty(E\cdot)$. Suppose that $s=(s^i)_{i\leq q}$ is a continuous $(-\infty, q)$-homotopy for the identity $A\cdot\subsetneq A\cdot$. Then one can construct for any $a>1$, a continuous $(-\infty, q)$-homotopy $\sigma=(\sigma^i)_{i\leq q}$ for the inclusion $A\cdot\{P[af]\}\subsetneq A\cdot\{P[f]\}$.

Theorem 2. *Let $f=(f_1, \cdots, f_\rho)$ be a ρ-tuple of $E\cdot$-analytic functions which are all $C\cdot$-admissible for a subcomplex $C\cdot$ of $C^\infty(E\cdot)$ and suppose that $s=(s^i)_{i\geq q}$ is a continuous (q, ∞)-homotopy of $C\cdot$ modulo $(0, 0)$. Then there is for $a>1$ a continuous (q, ∞)-homotopy $\sigma=(\sigma^i)_{i\geq q}$ of $C\cdot$ modulo $(C\cdot\{P(f)^c\}, C\cdot\{P(af)^c\})$ where $P(f)^c$ and $P(af)^c$ denote the complements of $P(f)$ and $P(af)$, respectively.*

The proofs of these theorems are both based on the Oka maps $M\ni z\mapsto(f_i(z), z)\in C$ $i=1, 2, \cdots, \rho$ having appeared implicitly in the notation $[V]_f$ in Definition 4. The method bears some resemblance to the proof of Theorem 2.7.6 in Hörmander's book [2].

The following corollaries will clarify how to apply the fundamental theorems.

Corollary 1. *Let $A\cdot$, s and f satisfy the hypothesis of Theorem 1. If A^i contains $C_0^\infty(E^i)$ for each $i\in Z$ and $P(f)$ is relatively compact in M, then the sequence*

$$\cdots\to C_0^\infty(P(f), E^{q-1})\to C_0^\infty(P(f), E^q)\to C_0^\infty(P(f), E^{q+1})$$

is exact.

Corollary 2. *Let $C\cdot$, s and f satisfy the hypothesis of Theorem 2. If A^i contains $C_0^\infty(E^i)$ for each $i\in Z$ and $P[bf]$ is compact for some $0<b<1$, then the sequence*

$$\Gamma(P[f], \underline{E}^{q-1})\to\Gamma(P[f], \underline{E}^q)\to\Gamma(P[f], \underline{E}^{q+1})\to\cdots$$

is exact, where $\underline{E}^i$, $i\in Z$ are the sheaves of germs of C^∞ sections of E^i.

In particular, if $f_1(z_0)=\cdots=f_\rho(z_0)=0$ and $P(bf)$, $0<b<\infty$ form a fundamental system of neighbourhoods of z_0, then the sequence of sheaves

$$\underline{E}^{q-1}\to\underline{E}^q\to\underline{E}^{q+1}\to\cdots$$

is exact at z_0.

    2. Application to tangential Cauchy-Riemann equation. First we shall define the Dolbeault sequence for a closed real submanifold M of a complex manifold X. (Note that the assumption of closedness is inessential because any submanifold of a manifold has always a neighbourhood in which it is closed.) Denote by I^0 the sheaf of germs of smooth functions in X which vanish when restricted to M, and by $\Omega^{(p,q)}$ the sheaf of germs of smooth (p, q) forms on X. Set further $\Omega=\Sigma\Omega^{(p,q)}$. Then Ω is a sheaf of rings by exterior multiplication. We denote by I the sheaf of ideals of Ω generated by $I^0+\bar\partial I^0$ where $\bar\partial: \Omega^{(p,q)}\to\Omega^{(p,q+1)}$ is the differentiation of the Dolbeault sequence for X. Following Kohn and Rossi [1] we now define sheaves $D^{(p,q)}$ so that the sequence $0\to I\to\Omega$

$\rightarrow D \rightarrow 0$ is exact where $D = \Sigma D^{(p,q)}$. (Note that I is homogeneous, that is, $I = \Sigma I^{(p,q)}$, $I^{(p,q)} = I \cap \Omega^{(p,q)}$.) The sheaf D has its support only on M, so D is regarded also as a sheaf over M. Since $\bar{\partial} I \subset I$, we can define $\bar{\partial}_b : D \rightarrow D$ such that $\bar{\partial}_b D^{(p,q)} \subset D^{(p,q+1)}$ making the following diagram commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \Omega & \longrightarrow & D & \longrightarrow & 0 \\
& & \bar{\partial}\downarrow & & \downarrow\bar{\partial} & & \downarrow\bar{\partial}_b & & \\
0 & \longrightarrow & I & \longrightarrow & \Omega & \longrightarrow & D & \longrightarrow & 0.
\end{array}
$$

Thus we have obtained for $p \geq 0$ the complex

$$
D^{(p,0)} \xrightarrow{\bar{\partial}_b} D^{(p,1)} \xrightarrow{\bar{\partial}_b} D^{(p,2)} \longrightarrow \cdots
$$

which we shall call *the p-th Dolbeault sequence for M*. The equation $\bar{\partial}_b u = 0$ for a section of D^0 is called usually the tangential Cauchy-Riemann equation of M. (For simplicity we write D^0 instead of $D^{(0,0)}$; note that D^0 is no other than the sheaf of germs of smooth functions over M.) We now assume that M is *generic*, that is, $T_z(X) = T_z(M) + \sqrt{-1}\, T_z(M)$ (not necessarily the direct sum) for every $z \in M$. (Here $T(M)$, $T(X)$ are the real tangent bundles of M, X respectively, and $T(M)$ is regarded as a subspace of the complex vector space $T_z(X)$.) Then the sheaves $D^{(p,q)}$ are all locally free D^0-sheaves and $\bar{\partial}_b$ is a first order differential operator. We shall denote by D°, the subbundle of $T(M)$ whose fiber D_z° is $T_z(M) \cap \sqrt{-1} T_z(M)$. We assume now that M is *of the second kind*, that is, any real vector field on M can be written locally as the sum of sections of D°, and of the brackets of sections of D°. Then we can define a surjective anti-symmetric bilinear map $[\ ,\ ]^z : D_z^\circ \times D_z^\circ \rightarrow W_z = T_z(M)/D_z^\circ$ so that for any $X, Y \in \Gamma(D^\circ)$, $[X_z, Y_z]^z$ is the residue class in W_z of the value $[X, Y]_z$ at z. Then it follows from the vanishing of the Nijenhuis tensor that

$$
[\sqrt{-1}x, \sqrt{-1}y]^z = [x, y]^z \qquad x, y \in D_z^\circ,
$$

so the map $(x, y) \mapsto [x, \sqrt{-1}y]^z$ is symmetric, and thus $f_\alpha(x, y)$, $x, y \in D_z^\circ$ is a symmetric bilinear form on D_z° for any $\alpha \in W_z^*$ if one sets $f_\alpha(x, y) = \langle \alpha, [x, \sqrt{-1}y] \rangle$.

Definition 6. Under the above notation, M is said to satisfy *condition ν_q* if and only if, for any $z \in M$ and for $0 \neq \alpha \in W_z^*$, one can find a subspace in D_z° of dimension at least $q+1$ on which f_α is positive definite.

Now we obtain the following theorem by applying Corollary 1 and Corollary 2.

Theorem 3. *Let M be a real submanifold of the second kind satisfying condition ν_{q_0} and let $\bar{q}_0 = \dim_C X - \operatorname{codim}_R M - q_0$. Then one*

can find for any $z \in M$, *an arbitrary small neighbourhood* Ω *of* z *such that the sequences*

$$0 \rightarrow \Gamma_c(\Omega, D^{(p,0)}) \rightarrow \cdots \rightarrow \Gamma_c(\Omega, D^{(p,q_0)}) \rightarrow \Gamma_c(\Omega, D^{(p,q_0+1)})$$

$$\Gamma(\overline{\Omega}, D^{(p,\tilde{q}_0-1)}) \rightarrow \Gamma(\overline{\Omega}, D^{(p,\tilde{q}_0)}) \rightarrow \cdots$$

are exact.　Here we have written $\Gamma_c(\Omega, D^{(p,q)})$ *for the set of sections of* $D^{(p,q)}$ *whose supports are compacts in* Ω.

This theorem is first proved for the standard real submanifolds in the sense of Tanaka [5].　The method is based on a priori estimates such as used in Hörmander [2] [3].　We pass from the standard case to the general, with the aid of a sub-ellipticity theorem concerning the Dolbeault sequences $D^{(p,\cdot)}$.　This theorem is a consequence from Theorem 1.4.2 of Hörmander [4]; it ensures the stability of the a priori estimates when approximating a given real submanifold by a suitable standard one.

The proofs of Theorems 1–3 will be published in Publ. Res. Inst. Math. Sci., Kyoto Univ., Vol. 8, No. 1.

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