210. *Topological Completions and Realcompactifications*

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Throughout this paper by a space we shall mean a completely regular T_1-space. The completion of a given space X with respect to its finest uniformity is called the topological completion of X, according to Morita [5], denoted by μX. The following question has been raised by Comfort [1]: Is there a locally compact space with a realcompactification which is not even a k-space? An ingenious example has been suggested by the referee of the above paper, and it is described in [1]. The cardinality of the space in this example is $\aleph_2$. Negrepontis [7] constructed further a locally compact separable space of cardinality $\aleph_1$, assuming the continuum hypothesis, whose realcompactification is a Lindelöf non k-space. In § 1, concerning this question, we shall point out the fact that if X is a normal space satisfying the condition $(cc \to c)$ and if Y is a subspace of υX such that $X \subsetneqq Y$, then Y is not a k-space (Theorem 1.1 below) and moreover investigate the relation between μX and Hewitt realcompactification υX of a given space X concerning local compactness (Theorem 1.5 below). In § 2, firstly we shall prove that the relation $\mu(X \times Y) = \mu X \times \mu Y$ holds whenever $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds. When we consider, in general, those pairs of spaces X and Y for which $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds, we are involved in their cardinalities deeply, and Comfort [1] obtained interesting results about this relation under certain conditions for cardinality of space. But we shall show that analogous theorems to Comfort's main results hold without regard to the cardinality in connection with the topological completion (Theorem 2.3 below). In § 3, we consider the classes of spaces which are defined in terms of the relation $\mu(X \times Y) = \mu X \times \mu Y$ similarly to McAuthur [4].

§ 1. The local compactness and k-ness of μX and υX.

In this section the following theorems are useful for our discussion of the relation between μX and υX.

(M.1.1) (Theorem 2, Morita [5]). $X \subset \mu X \subset \upsilon X$.

(C.1.1) (Theorem 4.6, Comfort [1]). In order that υX be locally compact, it is necessary and sufficient that for each $p \in \upsilon X$ there exist pseudocompact subsets A and B of X for which $p \in \mathrm{Cl}_{\upsilon X} A$ and there exists $f \in C^*(X)$ such that $f = 0$ on A and $f = 1$ on $X - B$.

(C.1.2) (Theorem 4 (Hager-Johnson), Comfort [1]). Let U be an open subset of X and suppose that $\mathrm{Cl}_{\upsilon X}\, U$ is compact. Then $\mathrm{Cl}_X\, U$ is pseudocompact.

X is said to *be topologically complete* if X has a complete uniformity. If any closed countably compact subset of X is always compact, then we shall call that X satisfies the *condition* $(cc\to c)$. For instance, if X satisfies any one of the following conditions, then X has $(cc\to c)$: 1) X is subparacompact (Theorem 3.5, Burke [2]), 2) X is symmetric (Corollary 2, Nedev [6]), 3) X is topologically complete.

Theorem 1.1. *Let X be a normal space satisfying the condition $(cc\to c)$. If Y is a subspace of υX such that $X \subsetneqq Y$, then Y is not a k-space.*

Proof. We shall firstly show that if there is a compact subset K of Y with $L = K\cap X \neq \emptyset$ and $K\cap(Y-X)\neq\emptyset$, then L must be compact. If L is pseudocompact, then since X is normal, L must be countably compact and hence L is compact by the condition $(cc\to c)$. Suppose that L is not pseudocompact. Then by the usual method there is a discrete family $\{U_n\}$ of open sets of X such that $x_n \in U_n \cap L$ for each n and there exists $f \in C(\beta X)$ such that $0 \leqq f \leqq 1$, $f(x_n)=1/n$, $f \geqq 1/n$ on U_n and $f=1$ on $X - \cup U_n$. K being compact, we have $\emptyset \neq K\cap Z(f) \subset \upsilon X - X$. This is impossible form the well known property of υX. From the above arguments, the intersection of a compact subset of Y with X is compact. Thus if Y is a k-space, then X is closed subspace of Y. This is impossible because X is dense in Y, and hence Y is not a k-space.

Corollary 1.2. *If X is normal topologically complete space, then any subspace Y of υX, $X\subsetneqq Y$, is not a k-space.*

Since it is known that $X \subsetneqq \upsilon X$ for a discrete space whose cardinal $|X|$ is measurable, we have the following from Corollary 1.2.

Corollary 1.3. *If X is a discrete space with $|X|$ which is measurable, then υX is not a k-space.*

If X is an M'-space, then μX is paracompact M-space (Theorem 2.5, Isiwata [3]). We have

Corollary 1.4. *If X is an M'-space and $\mu X \neq \upsilon X$, then υX is not a k-space.* (take $X=\mu X$ in Theorem 1.1)

Theorem 1.5. 1) *If μX is locally compact and $\mu X\neq \upsilon X$, then υX is not locally compact.*

2) *If υX is locally compact, then so is μX (and hence $\upsilon X = \mu X$ by 1)).*

Proof. 1) Suppose that υX is locally compact. Let U be a compact neighborhood of p, $p\in \upsilon X - \mu X$ with $U \subset \upsilon X$. Then $U\cap \mu X$ is pseudocompact by (C.1.2). Since μX is topologically complete, $U \cap \mu X$ must be compact. This implies that $p \in \mathrm{Cl}_{\upsilon X}(U\cap X) \subset \mu X$ which is impossible.

2) Let p be a point in $\mu X - X$. Since υX is locally compact, choose two sets A and B and $f \in C^*(X)$ in (C.1.1). Let F be a continuous extension of f over βX. We may assume that $A = F^{-1}(0) \cap X$. Since μX is topologically complete and A is pseudocompact, $\mathrm{Cl}_{\mu X} A$ is compact. If $p \notin \mathrm{Cl}_{\mu X} A$, then there is $g \in C(\beta X)$ with $g > 0$ on μX and $g(p) = 0$. Since $p \in \upsilon X$, this is impossible. Thus $p \in \mathrm{Cl}_{\mu X} A$. B being pseudocompact, $F^{-1}[0, 1/3] \cap X$ is pseudocompact and hence $F^{-1}[0, 1/3] \cap \mu X$ becomes to be a compact neighborhood (in μX) of p. Thus μX is locally compact.

§2. The relation $\mu(X \times Y) = \mu X \times \mu Y$.

Theorems used here are

(M.2.1) (Theorem 2.4, Morita [5]). μX is characterized as a space Y with the properties:

a) Y is topologically complete space containing X as a dense subspace.

b) Any continuous map f from X into an arbitrary metric space T can be extended to a continuous map from Y into T.

(M.2.2) (Theorem 5.1, Morita [5]). Let Y be a locally compact topologically complete space. Then the relation $\mu(X \times Y) = \mu X \times \mu Y$ holds for any space X.

Theorem 2.1. *If $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds, then we have $\mu(X \times Y) = \mu X \times \mu Y$.*

Proof. To prove this it is sufficient by (M.2.1) to show that any continuous map f from $X \times Y$ to any metric space T has a continuous extension over $\mu X \times \mu Y$. We shall firstly prove that f has a continuous extension over $X \times \mu Y$. The function $f_x = f \mid \{x\} \times Y \to T$ has a continuous extension $\tilde{f}_x$ over $\{x\} \times \mu Y$. Let q be any point in $\mu Y - Y$ and define the function $h_{x,q}(x, y) : X \times Y \cup (x, q) (= S) \to T$ by

$$h_{x,q}(x, y) = f(x, y) \qquad \text{for each } (x, y) \in X \times Y,$$
$$h_{x,q}(x, q) = \tilde{f}_x(x, q).$$

We define according to Comfort (p. 109, [1]) the following map F from $X \times \mu Y$ to T:

$$F = \cup \{h_{x,q} ; x \in X,\ q \in \mu Y - Y\},$$

that is, $F(x, q) = h_{x,q}(x, q)$ and $F \mid X \times Y = f$. Then if $h_{x,q}$ is continuous, then it is easy to see that F is continuous (notice that $X \times Y$ is dense in $X \times \mu Y$). Now suppose that $h_{x,q}$ is not continuous at (x, q). There is an open neighborhood W of the point $r = h_{x,q}(x, q)$ such that any open neighborhood U of (x, q) contains points which are not carried into W by $h_{x,q}$. This implies that $(x, q) \in \mathrm{Cl}_S\, h_{x,q}^{-1}(T - W)$. On the other hand, there exists a subset Y_1 of Y such that $q \in \mathrm{Cl}_Y Y_1$ and $h_{x,q}(\{x\} \times Y_1) \subset W_1$ where W_1 is an open set of T such that $r \in W_1 \subset \mathrm{Cl}_T W_1 \subset W$. Since T is a metric space, there is a continuous real-valued function k satisfying the conditions such that $k=0$ on $\mathrm{Cl}_T W_1$ and $k=1$ on $T-W$. It is obvious that $kf \in C(X\times Y)$. Since $\mu X\times \mu Y \subset \upsilon X\times \upsilon Y=\upsilon(X\times Y)$ by the assumption, kf has a continuous extension over $S(\subset X\times \upsilon Y)$. This leads that $kf=0$ on $\{x\}\times Y_1$ and $kf=1$ on $h_{x,q}^{-1}(T-W)$ which is impossible. Thus $h_{x,q}$ is continuous. Similarly the function F has a continuous extension over $\mu X\times \mu Y$. By (M.2.2) we have $\mu(X\times Y)=\mu X\times \mu Y$.

The following lemma is obvious.

Lemma 2.2. *Let f be a map from a k-space X to a metric space T. If f is continuous on every compact subset of X, then f is continuous on X.*

The followings are analogous theorems with respect to Hewitt realcompactification obtained by Comfort (Theorems 2.4, $\cdots$, [1]). But in our theorem there is no need to consider "nonmeasurability" or "measurability" of the cardinal number of spaces.

Theorem 2.3. *Let μX be locally compact and Y be a k-space. Then we have $\mu(X\times Y)=\mu X\times \mu Y$.*

Proof. Let T be a metric space and let f be a continuous map of $X\times Y$ to T. We shall show that f has a continuous extension over $\mu X\times \mu Y$. $f_y=f\,|\,X\times\{y\}$ being continuous, f_y has a continuous extension $\tilde{f}_y : \mu X\times\{y\}\to T$ by (M.2.1). Put $F=\cup\{\tilde{f}_y\,;\,y\in Y\}$. Then F is the map of $\mu X\times Y$ to T. Since the product of a locally compact space with a k-space is a k-space, $\mu X\times Y$ is a k-space. Thus in order to show that the continuity of F, we need only show that the restriction of F to each compact subset K of $\mu X\times Y$ is continuous by Lemma 2.2. Let π be the projection of $\mu X\times Y$ onto Y and let us put $F_K=F\,|\,\mu X\times \pi K$. By the same method in the proof of (M.2.2), F_K is continuous because K is compact and $F_K\,|\,X\times \pi K=f\,|\,X\times\pi K$. This implies that F is a continuous extension of f over $\mu X\times Y$. By (M.2.2) we have $\mu(\mu X\times Y)=\mu X\times \mu Y$ and hence F has a continuous extension over $\mu X\times\mu Y$. Thus the relation $\mu(X\times Y)=\mu X\times \mu Y$ follows from (M.2.2).

The pseudocompactness of X implies the compactness of μX (Theorem 3.1, Morita [5]), and hence we have

Corollary 2.4. *Let X be pseudocompact and Y be a k-space. Then the relation $\mu(X\times Y)=\mu X\times\mu Y$ holds.*

Since an essential part of the proof of Theorem 2.3 is that $\mu X\times Y$ and $\mu X\times \mu Y$ are k-spaces, we have the following

Corollary 2.5. *If $\mu X\times Y$ and $\mu X\times \mu Y$ are k-spaces, then we have $\mu(X\times Y)=\mu X\times\mu Y$.*

Corollary 2.6. *If X is an M'-space, then we have $\mu(X\times Y)=\mu X\times \mu Y$ for any paracompact M-space Y.*

This follows from the facts that for an M'-space X, μX is a paracompact M-space and the product of paracompact M-spaces is a paracompact M-space (notice that a paracompact M-space is a k-space).

Since X is an M'-space if and only if μX is a paracompact M-space (Theorem 4.4, Morita [5]) we have

Corollary 2.7. *If X is an M'-space, then $X\times Y$ is an M'-space for any paracompact M-space Y.*

§3. The classes $\mathcal{R}(\mu)$, $\mathcal{M}(\mu)$ and $\mathcal{P}(\mu)$.

In this section we shall show the results obtained by McAuthur (§§3, 5, in [4]) can be extended to the case of topological completion instead of Hewitt realcompactification, by changing definition of the property Ω and using the results of Morita. The arguments used in [4] passes current everywhere in our discussion in this section.

Let Φ be the finest uniformity of X. Let $\mathcal{A}(X)$ be the set consisting of all pair (T, f) where T is any metric space and f is any continuous map of X to T. A filter base $\mathcal{F}$ on X is said to have property $\Omega(X, \mu)$ if every (T, f) in $\mathcal{A}(X)$ and every $\varepsilon>0$, there is a set F in $\mathcal{F}$ with the diameter of $f(F)$ is less than ε. It is easy to see that a filter base $\mathcal{F}$ has the property $\Omega(X, \mu)$ if and only if it is a Cauchy filter base with respect to Φ. A pair of space (X, Y) is said to have the *rectangle condition* if whenever $\mathcal{F}$ (resp. $\mathcal{G}$) is a filter base on X (resp. Y) with the property $\Omega(X, \mu)$ (resp. $\Omega(Y, \mu)$) then the filter base $\mathcal{F}\times\mathcal{G}=\{F\times G; F\in\mathcal{F}, G\in\mathcal{G}\}$ has the property $\Omega(X\times Y, \mu)$.

3.1. *$\mu(X\times Y)=\mu X\times\mu Y$ if and only if the pair (X, Y) satisfies the rectangle condition.*

The proof of this is the same as in the proof of proposition 3.3 in [4] except using (M.2.1).

Let $\mathcal{R}(\mu)$ be the class of all spaces X such that for every space Y the relation $\mu(X\times Y)=\mu X\times\mu Y$ holds. By (M.2.2) $\mathcal{R}(\mu)$ contains the class of locally compact topologically complete spaces. We have moreover the following proposition whose proof is also the same as in the proof of Theorem 5.2 in [4].

3.2. *If X is a member of $\mathcal{R}(\mu)$, then X is topologically complete.*

Let $\mathcal{M}(\mu)$ be the class of all spaces X such that the relation $\mu(X\times Y)=\mu X\times\mu Y$ holds for every topologically complete space Y. Remarking the fact that for discrete space Y and a point p in $\beta Y-Y$, $Y\cup\{p\}$ with the relative topology is always paracompact, we have the following theorem similarly to Theorem 5.5 in [4].

3.3. Theorem. *X is a member of $\mathcal{M}(\mu)$ if and only if X is topologically complete.*

Let $\mathcal{P}(\mu)$ be the class of all spaces X such that the relation $\mu(X\times\mu X)=\mu X\times\mu X$ holds. By Corollary 2.6 and Theorem 3.2, $\mathcal{P}(\mu)$ contains the class of topologically complete spaces and the class of

M'-spaces.

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