# 209. *Hypersurfaces of a Euclidean Space $R^{4m}$*

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**Introduction.** K. Yano and M. Okumura [5] have shown that the existence of the so called $(f, g, u, v, \lambda)$-structure on hypersurfaces of an almost contact manifold and on submanifolds of codimension 2 of an almost Hermitian manifold.

D. E. Blair, G. D. Ludden and K. Yano [1] have studied complete hypersurfaces immersed in $S^{2n+1}$ and showed that (1) if the Weingarten map of the immersion and $f$ commute then the hypersurface is a sphere, and (2) if the Weingarten map $K$ of the immersion and $f$ satisfy $fK+Kf=0$ and the hypersurface is of constant scalar curvature, then it is a great sphere or $S^n \times S^n$.

On the other hand, Y. Y. Kuo [2] has shown the existence of an almost contact 3-structure on $R^{4m+3}$ and that of a Sasakian 3-structure on $S^{4m+3}$ and on the real projective space $P^{4m+3}$.

The main purpose of this paper is, after showing that an orientable hypersurface of a Hermitian manifold with quaternion structure admits an almost contact 3-structure $(\phi_i, \xi_i, \eta_i)$, $i=1, 2, 3$, to classify complete hypersurfaces of $R^{4m}$ satisfying $\phi_i H - H\phi_i = 0$, $i=1, 2, 3$ and those satisfying $\phi_i H + H\phi_i = 0$, $i=1, 2, 3$. The results are:

**Theorem 1.** *Let $N$ be a complete hypersurface of $R^{4m}(m \geqq 2)$. If the Weingarten map of the immersion and $\phi_i$, $i=1, 2, 3$ commute, then $N$ is one of the following*

(i) *a hyperplane,*

(ii) *a sphere,*

(iii) $R^{4t} \times S^{4s+3}$, $t+s=m-1$, $t \geqq 1$, $s \geqq 0$.

**Theorem 2.** *Let $N$ be a complete hypersurface of $R^{4m}(m \geqq 1)$. If the Weingarten map $H$ of the immersion and $\phi_i$ satisfy $\phi_i H + H\phi_i = 0$, then it is a hyperplane.*

For the case $m=1$ in Theorem 1, we have, as a corollary,

**Corollary.** *Let $N$ be a complete hypersurface of $R^4$. If the Weingarten map of the immersion and $\phi_i$, $i=1, 2, 3$ commute, then $N$ is either a hyperplane or a sphere.*

**1. Preliminaries.** First, let $M=M^{4m}$ be a differentiable manifold with quaternion structure $(\Phi_1, \Phi_2)$, where a quaternion structure is, by definition, a pair of two almost complex structures $\Phi_1$, $\Phi_2$ such that

$$\Phi_1\Phi_2 + \Phi_2\Phi_1 = 0. \tag{1}$$

It is known that there exists a Riemannian metric $G$ such that

(2) $$G(\Phi_1 X, \Phi_1 Y)=G(\Phi_2 X, \Phi_2 Y)=G(X, Y).$$

We call a manifold with $\Phi_1$, $\Phi_2$ and $G$ satisfying (2) a *Hermitian manifold with quaternion structure.* If, furthermore, $G$ is Kaehlerian with respect to both $\Phi_1$ and $\Phi_2$, such a manifold is called a *Kaehlerian manifold with quaternion structure.* $R^{4m}$ is an example of a Kaehlerian manifold with quaternion structure. If we put $\Phi_3=\Phi_1\Phi_2$, then $\Phi_3$ is also an almost complex structure and $\Phi_i$, $i=1,2,3$ satisfy

(3) $$\Phi_i\Phi_j=-\Phi_j\Phi_i=\Phi_k,$$

where $(i, j, k)$ is any cyclic permutation of $(1,2,3)$.

Secondly, let $N=N^{4n+3}$ be a differentiable manifold with an almost contact 3-structure $(\Phi_i, \xi_i, \eta_i)$, $i=1,2,3$, where an almost contact structure is, by definition, a pair of three almost contact structure $(\phi_i, \xi_i, \eta_i)$, $i=1,2,3$ satisfying

(4) $$\begin{cases}\eta_i(\xi_j)=\eta_j(\xi_i)=0,\\ \phi_i\xi_j=-\phi_j\xi_i=\xi_k,\\ \eta_i\circ\phi_j=-\eta_j\circ\phi_i=\eta_k,\\ \phi_i\phi_j-\xi_i\otimes\eta_j=-\phi_j\phi_i+\xi_j\otimes\eta_i=\phi_k,\end{cases}$$

for any cyclic permutation $(i, j, k)$ of $(1,2,3)$.
There exists a Riemannian metric $g$ such that

(5) $$g(\xi_i, X)=\eta_i(X),$$

(6) $$g(\phi_i X, \phi_i Y)=g(X, Y)-\eta_i(X)\eta_i(Y),$$

$(i=1,2,3)$, for any vectors $X$ and $Y$. This metric is called an *associated metric of the* 3-*structure.* If, furthermore, $\xi_i$ $(i=1,2,3)$ are mutually orthogonal Sasakian structure, such a structure is called a *Sasakian* 3-*structure.*

**2. Hypersurfaces of a Hermitian manifold with quaternion structure.** Let $M=M^{4m}$ be a Hermitian manifold with quaternion structure $(\Phi_i, G)$, $i=1,2,3$, $N=N^{4m-1}$ be an orientable hypersurface of $M$ and $\pi: N\to M$ be its imbedding. If we put

(7) $$g(X, Y)=G(\pi_* X, \pi_* Y),$$

then $g$ is a Riemannian metric on $N$.

Let $C$ be a field of unit normals defied on $\pi(N)$ and put $(\phi_i, \xi_i, \eta_i)$, $i=1,2,3$ by

(8) $$\Phi_i\pi_* X=\pi_*\phi_i X+\eta_i(X)C,$$

(9) $$\Phi_i C=-\pi_*\xi_i,$$

then we can easily see that $(\phi_i, \xi_i, \eta_i)$, $i=1,2,3$ satisfy (4) and $g$ satifies (5) and (6). Thus, we have

**Proposition 1.** *An orientable hypersurface $N$ of a Hermitian manifold with quaternion structure admits an almost contact* 3-*structure and the natually induced metric $g$ on $N$ is an associated metric of the above almost contact* 3-*structure.*

Now, we assume further that $M$ is a Kaehlerian manifold with quaternion structure. We put

$$\tilde{\nabla}_{\pi_*X}\pi_*Y=\pi_*\nabla_X Y+h(X,Y)C, \tag{10}$$

$$\tilde{\nabla}_{\pi_*X}C=-\pi_*HX, \tag{11}$$

where $\tilde{\nabla}$ is the Kaehlerian connection of $G$, $h(X, Y)$ is the second fundamental form and $H$ is the corresponding Weingarten map.
Calculating both $\tilde{\nabla}_{\pi_*X}\Phi_i\pi_*Y$ and $\tilde{\nabla}_{\pi_*X}\Phi_i C$ in two ways, we have

$$\begin{aligned}\tilde{\nabla}_{\pi_*X}\Phi_i\pi_*Y&=\pi_*\phi_i\nabla_X Y+\eta_i(\nabla_X Y)C-h(X,Y)\pi_*\xi_i\\&=\pi_*[(\nabla_X\phi_i)Y+\phi_i\nabla_X Y-\eta_i(Y)HX]+((\nabla_X\eta_i)(Y)+\eta_i(\nabla_X Y)\\&\quad+h(X,\phi_i Y))C,\end{aligned}$$

$$\begin{aligned}\tilde{\nabla}_{\pi_*X}\Phi_i C&=-\pi_*\phi_i HX-\eta_i(HX)C\\&=-\pi_*\nabla_X\xi_i-h(X,\xi_i)C,\end{aligned}$$

from which we have

$$(\nabla_X\phi_i)Y=\eta_i(Y)HX-h(X,Y)\xi_i, \tag{12}$$

$$(\nabla_X\eta_i)(Y)=-h(X,\phi_i Y), \tag{13}$$

$$\nabla_X\xi_i=\phi_i HX. \tag{14}$$

The following lemmas are needed later.

**Lemma 2.** *If $H\phi_i=\phi_i H$, $i=1, 2, 3$, then $\xi_1$, $\xi_2$ and $\xi_3$ are the characteristic vectors of $H$ and the corresponding characteristic roots are the same, that is we have*

$$H\xi_i=\lambda\xi_i \qquad (i=1,2,3), \tag{15}$$

*for some scalar $\lambda$.*

**Proof.** By assumption, we may put $H\xi_i=\lambda_i\xi_i$ $(i=1,2,3)$. Thus, using (4) and (6) we have

$$\begin{aligned}\lambda_i&=g(H\xi_i,\xi_i)=g(\phi_k H\xi_i,\phi_k\xi_i)+\eta_k(H\xi_i)\eta_k(\xi_i)\\&=g(H\phi_k\xi_i,\phi_k\xi_i)\\&=g(H\xi_j,\xi_j)\\&=\lambda_j.\end{aligned}$$

q.e.d.

**Lemma 3.** *If $H\phi_i=-\phi_i H$, $i=1, 2, 3$, then $\xi_1$, $\xi_2$ and $\xi_3$ are the characteristic vectors of $H$ corresponding to the characteristic root 0.*

**Proof.** As in Lemma 2, we may put $H\xi_i=\mu_i\xi_i$, $i=1, 2, 3$. Then we have

$$\begin{aligned}\mu_i=g(H\xi_i,\xi_i)&=g(\phi_k H\xi_i,\phi_k\xi_i)+\eta_k(H\xi_i)\eta_k(\xi_i)\\&=-g(H\phi_k\xi_i,\phi_k\xi_i)\\&=-g(H\xi_j,\xi_j)\\&=-\mu_j,\end{aligned}$$

which implies $\mu_i=0$, $(i=1,2,3)$. q.e.d.

**3. Proofs of Theorems.** Let $N$ be an orientable hypersurface of $R^{4m}$. Hereafter we use the same notations which were used in the previous section by identifying $R^{4m}$ with $M$. Then the Codazzi equation of the hypersurface can be given by

$$(\nabla_X H)Y=(\nabla_Y H)X. \tag{16}$$

**Proof of Theorem 1.** Setting $Y$ equal to $\xi_i$ in (16), we have

(17) $$(\nabla_X H)\xi_i = (\nabla_{\xi_i} H)X.$$

But, since

$$\begin{aligned}(\nabla_X H)\xi_i &= \nabla_X H\xi_i - H\nabla_X \xi_i \\ &= (\nabla_X \lambda)\xi_i + \lambda \nabla_X \xi_i - H\nabla_X \xi_i \qquad \text{(by (15))} \\ &= (\nabla_X \lambda)\xi_i + \lambda\phi_i HX - H\phi_i HX \qquad \text{(by (14))} \\ &= (\nabla_X \lambda)\xi_i + \lambda\phi_i HX - \phi_i H^2 X,\end{aligned}$$

we have

(18) $$(\nabla_{\xi_i} H)X = (\nabla_X \lambda)\xi_i + \lambda\phi_i HX - \phi_i H^2 X.$$

Setting $X$ equal to $\xi_k$ in (18), we have

$$\begin{aligned}(\nabla_{\xi_i} H)\xi_k &= (\nabla_{\xi_k}\lambda)\xi_i + \phi_i H\xi_k - \phi_i H^2 \xi_k \\ &= (\nabla_{\xi_k}\lambda)_i + \lambda^2\phi_i\xi_k - \lambda^2\phi_i\xi_k \\ &= (\nabla_{\xi_k}\lambda)\xi_i.\end{aligned}$$

Since $(\nabla_{\xi_i} H)\xi_k = (\nabla_{\xi_k} H)\xi_i$ by the Codazzi equation, we have

$$(\nabla_{\xi_k}\lambda)\xi_i = (\nabla_{\xi_i}\lambda)\xi_k,$$

which implies

(19) $$\nabla_{\xi_i}\lambda = 0,$$

(20) $$(\nabla_{\xi_i} H)\xi_k = 0.$$

Therefore, taking account of $\nabla_{\xi_i}\xi_i = \phi_i H\xi_i = \lambda\phi_i\xi_i = 0$, we have from (18)

$$\begin{aligned}\nabla_X \lambda &= g((\nabla_{\xi_i} H)X, \xi_i) \\ &= g(\nabla_{\xi_i} HX, \xi_i) - g(H\nabla_{\xi_i} X, \xi_i) \\ &= \nabla_{\xi_i}(g(HX, \xi_i)) - g(\nabla_{\xi_i} X, H\xi_i) \\ &= \lambda g(\nabla_{\xi_i} X, \xi_i) - \lambda g(\nabla_{\xi_i} X, \xi_i) \qquad \text{(by (19))} \\ &= 0.\end{aligned}$$

Hence $\lambda$ is constant and consequently (18) reduces to

(21) $$(\nabla_{\xi_i} H)X = \lambda\phi_i HX - \phi_i H^2 X.$$

Let $\{e_s, \phi_1 e_s, \phi_2 e_s, \phi_3 e_s, \xi_1, \xi_2, \xi_3\}$ be an orthonormal basis which diagonalizes $H$. We denote the principal curvature corresponding to $e_s$ by $\alpha_s$ that is also the principal curvature corresponding to $\phi_1 e_s$, $\phi_2 e_s$ and $\phi_3 e_s$, since $H\phi_i = \phi_i H$, $i = 1, 2, 3$. Consider $\nabla_{\xi_1} H$ as a tensor of type (1,1) on $N$. By (20), $\xi_i$, $i = 1, 2, 3$ are characteristic vectors corresponding to the characteristic root 0. Let $X = \sum_{s=1}^{m} (a_s e_s + b_s\phi_1 e_s + c_s\phi_2 e_s + d_s\phi_3 e_s)$ be a characteristic vector of $\nabla_{\xi_1} H$ other than $\xi_1$, $\xi_2$ and $\xi_3$. Let $\beta$ be its corresponding characteristic root. Then we have $(\nabla_{\xi_1} H)X = \beta X$. But the left hand side can be calculated as

$$\begin{aligned}(\nabla_{\xi_1} H)X &= \phi_1 H \sum_s (a_s e_s + b_s\phi_1 e_s + c_s\phi_2 e_s + d_s\phi_3 e_s) \\ &\quad - \phi_1 H^2 \sum_s (a_s e_s + b_s\phi_1 e_s + c_s\phi_2 e_s + d_s\phi_3 e_s) \\ &= \sum_s \{\alpha_s(\lambda - \alpha_s)a_s\phi_1 e_s - \alpha_s(\lambda - \alpha_s)b_s e_s + \alpha_s(\lambda - \alpha_s)c_s\phi_3 e_s \\ &\quad - \alpha_s(\lambda - \alpha_s)d_s\phi_2 e_s\},\end{aligned}$$

by virtue of (21) with $i = 1$ and (4).

Thus, comparing the coefficients of $e_s$, $\phi_1 e_s$, $\phi_2 e_s$ and $\phi_3 e_s$ of the above equation we have

$$\begin{cases} \beta a_s + \alpha_s(\lambda-\alpha_s)b_s = 0, \\ \alpha_s(\lambda-\alpha_s)a_s - \beta b_s = 0, \\ \beta c_s + \alpha_s(\lambda-\alpha_s)d_s = 0, \\ \alpha_s(\lambda-\alpha_s)c_s - \beta d_s = 0. \end{cases}$$

Since $X \neq 0$, we must have

$$\beta^2 + \alpha_s^2(\lambda-\alpha_s)^2 = 0$$

from the theory of a system of linear equations.
Hence we have $\beta=0$ and consequently (21) with $i=1$ reduces to

$$\lambda\phi_1 HX - \phi_1 H^2 X = 0, \tag{22}$$

for any vector $X$ on the hypersurface.
Therefore, putting $X=e_s$, $s=1, \cdots, m-1$, we have $\alpha_s(\lambda-\alpha_s)=0$, which shows that the hypersurface has distinct principal curvatures at most two and they are constant. There are three possibilities: if $\lambda=0$, then all $\alpha_s$ are automatically equal to 0 and the hypersurface is totally geodesic thereby it is a hyperplane. If $\lambda \neq 0$ and none of $\alpha_s$ are 0, then all $\alpha_s$ are equal to $\lambda$ and the hypersurface is totally umbilical thereby it is a sphere. The last possibility gives that the hypersurface is $R^{4t} \times S^{4s+3}$, $t+s=m-1$, $t \geqq 1$, $s \geqq 0$ by the same argument as in [4], which completed the proof.

**Proof of Theorem 2.** We have

$$\begin{aligned} \nabla_X H\phi_i Y &= (\nabla_X H)\phi_i Y + H(\nabla_X \phi_i)Y + H\phi_i \nabla_X Y \\ &= (\nabla_X H)\phi_i Y + H(\eta_i(Y)HX - h(X, Y)\xi_i) + H\phi_i \nabla_X Y \qquad \text{(by (12))} \\ &= (\nabla_X H)\phi_i Y + \eta_i(Y)H^2X + H\phi_i \nabla_X Y \qquad \text{(since } H\xi_i = 0). \end{aligned}$$

But, since $H\phi_i Y = -\phi_i HY$, we have

$$\begin{aligned} \nabla_X H\phi_i Y &= -\nabla_X \phi_i HY \\ &= -(\nabla_X \phi_i)HY - \phi_i(\nabla_X H)Y - \phi_i H \nabla_X Y \\ &= -(\eta_i(HY)HX - h(X, HY)\xi_i) - \phi_i(\nabla_X H)Y - \phi_i H\nabla_X Y \\ &= g(HX, HY) - \phi_i(\nabla_X H)Y - \phi_i H \nabla_X Y. \end{aligned}$$

Hence we have

$$(\nabla_X H)\phi_i Y + \eta_i(Y)H^2X = g(HX, HY)\xi_i - \phi_i(\nabla_X H)Y.$$

Thus we have

$$g((\nabla_X H)\phi_i Y, \xi_i) = g(HX, HY). \tag{23}$$

But we have

$$\begin{aligned} g((\nabla_X H)\phi_i Y, \xi_i) &= g((\nabla_{\phi_i Y} H)X, \xi_i) \qquad \text{(by (16))} \\ &= \nabla_{\phi_i Y}(g(HX, \xi_i)) - g(HX, \nabla_{\phi_i Y}\xi_i) \\ &= -g(HX, \phi_i H\phi_i Y) \\ &= g(HX, \phi_i^2 HY) \qquad \text{(since } H\phi_i = -\phi_i H) \\ &= -g(HX, HY) \qquad \text{(by (4))}, \end{aligned}$$

which, together with (23), implies $H=0$ and hence the hypersurface is totally geodesic thereby it is a hyperplane.

## References

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