208. *A Remark on Nowhere Differentiability of Sample Functions of Gaussian Processes*

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1. The Result. Let $\{X(t,\omega);0\le t\le 1\}$ be a Gaussian process with the mean $E[X(t)]=0$. We assume in this paper that the process has continuous sample functions and there exists an even, non-negative, and non-decreasing function $\varphi(h)$ such that

$$E[(X(t+h)-X(t))^2]\ge\varphi^2(h).$$

The sample functions of the Brownian motion are, as well known, nowhere differentiable with probability one [5]. J. Yeh [6] proved that the sample functions are almost nowhere differentiable with probability one, for the Gaussian processes satisfying the condition $\lim_{h\downarrow 0} h/\varphi(h)=0$. But there exists a gap between the property of almost nowhere differentiability with probability one and the property of nowhere differentiability with probability one, because almost nowhere differentiability is essentially non differentiability at each point with probability one, but this does not yield nowhere differentiability with probability one.

In this paper we shall prove a theorem about nowhere differentiability of sample functions as follows:

Theorem. *If there exists a positive integer q such that*

$$\lim_{h\downarrow 0}\left(\frac{h}{\varphi(h)}\right)^q\frac{1}{h}=0,$$

and if there exists a positive integer p such that

$$\overline{\lim_{h\downarrow 0}}\sup_{|t-s|\ge ph}|E[\varDelta_h Y(t)\varDelta_h Y(s)]|\le 1/(2q),$$

where

$$\varDelta_h Y(t)=(X(t+h)-X(t))/\{E[(X(t+h)-X(t))^2]\}^{1/2},$$

then we have

$$P\left[\overline{\lim_{h\downarrow 0}}\frac{|X(t+h)-X(t)|}{h}=+\infty\qquad\text{for all } t,\ 0\le t<1\right]=1.$$

Recently, S. M. Berman has proved nicely the same fact for some class of Gaussian processes by means of local times of the stochastic processes [1][2]. Our class which satisfies the conditions of Theorem is not contained in Berman's class, and the proof of our Theorem is much simpler, making use of the idea of [3].

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2. Proof of Theorem. Set,

$$A(t, K, a)=\{\omega\,;\,|X(t+h)-X(t)|\leq Kh \qquad \text{for all } 0\leq h\leq 1/a\},$$

$$A_{i,k}^{(n)}=\left\{\omega\,;\,\left|X\left(\frac{i+pk+1}{a_n}\right)-X\left(\frac{i+pk}{a_n}\right)\right|\leq 2K(pq+2)/a_n\right\},$$

$k=0, 1, 2, \cdots, q-1$, $i=0, 1, 2, \cdots, a_n-pq+p-1$, $a_n=$integer,

and

$$B_i^{(n)}=\bigcap_{k=0}^{q-1} A_{i,k}^{(n)}.$$

Then we have

$$\left\{\omega\,;\,\overline{\lim_{h\downarrow 0}}\frac{|X(t+h)-X(t)|}{h}<K \qquad \text{for some } t \text{ in } [0, 1)\right\}$$

$$\subset\bigcup_{0\leq t<1}\bigcup_{a_n\uparrow\infty} A(t, K, a_n)$$

$$\subset\bigcap_{m\geq 1}\bigcup_{n\geq m}\left(\bigcup_i B_i^{(n)}\right).$$

Now we shall estimate $P(B_i^{(n)})$;

$$P(B_i^{(n)})=F_{A_i^{(n)}}\left(\frac{2K(pq+2)}{a_n\varphi(1/a_n)}\right),$$

where

$$F_A(a)=\frac{1}{(2\pi)^{q/2}\sqrt{\det A}}\int\cdots\int_{|x_k|\leq a}\exp\left\{-\frac{1}{2}(A^{-1}x, x)\right\}dx_1\cdots dx_q,$$

matrix $A_i^{(n)}=(r_i^{(n)}(k, k'))_{k,k'=0}^{q-1}$, and its element

$$r_i^{(n)}(k, k')=E\left[\varDelta_{1/a_n}Y\left(\frac{i+pk}{a_n}\right)\varDelta_{1/a_n}Y\left(\frac{i+pk'}{a_n}\right)\right].$$

By virtue of our assumptions, it follows that

$$r_i^{(n)}(k, k)=1 \quad \text{and} \quad |r_i^{(n)}(k, k')|\leq 1/(2q), \qquad (k\neq k')$$

for sufficiently large n. Since, for sufficiently large n, there exists a positive constant c_1 independent of n such that $\det A_i^{(n)}\geq c_1$, it holds that

$$P(B_i^{(n)})\leq\frac{\text{const.}}{(a_n\varphi(1/a_n))^q}.$$

Thus we have

$$P\left(\bigcup_i B_i^{(n)}\right)\leq\frac{\text{const. } a_n}{(a_n\varphi(1/a_n))^q},$$

which tends to zero as $n\to\infty$. Since we could choose an increasing sequence $\{a_n\}$ such that

$$\sum_{n=1}^{\infty}\frac{a_n}{(a_n\varphi(1/a_n))^q}<+\infty,$$

we have, owing to Borel-Cantelli lemma,

$$P\left(\overline{\lim_{n\to\infty}}\bigcup_i B_i^{(n)}\right)=0.$$

Finally, setting $K=K_n\uparrow+\infty$, we have the proof of Theorem.

3. Example. Assume that the process has stationary increment and we set $E[((X(t)-X(s))^2]=\sigma^2(|t-s|)$. Then it is easily checked that our Theorem is valid for the following cases;

(i) $\sigma(h)=h^\alpha$, $0<\alpha<1$,

(ii) $\sigma(h)$ is a regular varying function with exponent $0<\alpha<1/2$,

(iii) $\sigma(h)$ is a slowly varying function,

(iv) $\sigma(h)$ is a nearly slowly varying function with exponent $0<\alpha<1/2$ and $\sigma^2(h)$ is concave near the origin (see [4]).

References

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