202. On the Asymptotic Behavior of Solutions of Certain Third Order Ordinary Differential Equations

By Tadayuki HARA
Osaka University

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1. Introduction. Our purpose here is to study the behavior as $t\to\infty$ of solutions of the differential equations

(1.1) $$\dddot{x}+a(t)\ddot{x}+b(t)\dot{x}+c(t)x=e(t) \qquad \left(\dot{x}=\frac{dx}{dt}\right),$$

(1.2) $$\dddot{x}+a(t)\ddot{x}+b(t)\dot{x}+c(t)h(x)=e(t),$$

(1.3) $$\dddot{x}+a(t)f(x,\dot{x})\ddot{x}+b(t)g(x,\dot{x})\dot{x}+c(t)h(x)=e(t).$$

We assume the following conditions throughout this note.

(c_1) $a(t)$, $b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty)$.

(c_2) $e(t)$ is continuous and absolutely integrable on $[0,\infty)$.

(c_3) $h(x)$ is continuously differentiable and real-valued for all x.

(c_4) $f(x,y)$, $f_x(x,y)$, $g(x,y)$ and $g_x(x,y)$ are continuous and real-valued for all (x,y).

In [2], the author considered the conditions under which all solutions of the non-autonomous equations (1.1) and (1.3) with $e(t)\equiv 0$ and $h(x)=x$ tend to zero as $t\to\infty$.

2. Theorems.

Theorem 1. *Suppose that $a(t)$, $b(t)$ and $c(t)$ are continuously differentiable and $e(t)$ is continuous on $[0,\infty)$ and following conditions are satisfied;*

(i) $A\geqq a(t)\geqq a_0>0$, $B\geqq b(t)\geqq b_0>0$, $C\geqq c(t)\geqq c_0>0$ *for* $t\in[0,\infty)$,

(ii) $xh(x)>0$ $(x\neq 0)$, $H(x)=\int_0^x h(\xi)d\xi\to+\infty$ *as* $|x|\to\infty$,

(iii) $\dfrac{a_0b_0}{C}>h_1\geqq h'(x)$,

(iv) $\mu a'(t)+b'(t)-\dfrac{1}{\rho}c'(t)<\dfrac{a_0b_0-Ch_1}{2} \qquad \left(\mu=\dfrac{a_0b_0+Ch_1}{2b_0},\ \rho=\dfrac{\mu}{h_1}\right)$,

(v) $\int_0^\infty |c'(t)|\,dt<\infty$, $c'(t)\to 0$ *as* $t\to\infty$,

(vi) $\int_0^\infty |e(t)|\,dt<\infty$.

Then every solution $x(t)$ of (1.2) *is uniform-bounded and satisfies $x(t)\to 0$, $\dot{x}(t)\to 0$, $\ddot{x}(t)\to 0$ as $t\to\infty$.*

Corollary 1. *Suppose that the conditions* (i), (v), (vi) *and in addi-*

tion following conditions are satisfied;

(iii)′ $a_0b_0-C>0$,

(iv)′ $\mu a'(t)+b'(t)-\frac{1}{\mu}c'(t)<\frac{a_0b_0-C}{2} \qquad \left(\mu=\frac{a_0b_0+C}{2b_0}\right)$.

Then every solution $x(t)$ *of* (1.1) *is uniform-bounded and satisfies* $x(t)\to 0$, $\dot{x}(t)\to 0$, $\ddot{x}(t)\to 0$ *as* $t\to\infty$.

Theorem 2. *Suppose that* $a(t)$, $b(t)$ *and* $c(t)$ *are continuously differentiable and* $e(t)$ *is continuous on* $[0,\infty)$ *and following conditions are satisfied*;

(i) $A\geqq a(t)\geqq a_0>0$, $B\geqq b(t)\geqq b_0>0$, $C\geqq c(t)\geqq c_0>0$ *for* $t\in[0,\infty)$,

(ii) $f(x,y)\geqq f_0>0$, $yf_x(x,y)\leqq 0$ *for all* (x,y),

(iii) $g(x,y)\geqq g_0>0$, $yg_x(x,y)\leqq 0$ *for all* (x,y),

(iv) $xh(x)>0\ (x\neq 0)$, $H(x)=\int_0^x h(\xi)d\xi\to+\infty$ *as* $|x|\to\infty$,

(v) $\frac{a_0b_0f_0g_0}{C}>h_1\geqq h'(x)$,

(vi) $\frac{|a'(t)|}{a_0}\mu^2+\frac{|b'(t)|}{b_0}b(t)g_0-c'(t)\frac{b(t)}{c(t)}g_0<\frac{a_0b_0f_0g_0-Ch_1}{2}$

$$\left(\mu=\frac{a_0b_0f_0g_0+Ch_1}{2b_0g_0}\right),$$

(vii) $\int_0^\infty|a'(t)|\,dt<\infty$, $\int_0^\infty|b'(t)|\,dt<\infty$, $\int_0^\infty|c'(t)|\,dt<\infty$, $c'(t)\to 0$ *as* $t\to\infty$,

(viii) $\int_0^\infty|e(t)|\,dt<\infty$.

Then every solution $x(t)$ *of* (1.3) *is uniform-bounded and satisfies* $x(t)\to 0$, $\dot{x}(t)\to 0$, $\ddot{x}(t)\to 0$ *as* $t\to\infty$.

3. Auxiliary Lemmas.

Consider a system of differential equations

$$\dot{X}=F(t,X)+G(t,X) \tag{3.1}$$

where $F(t,X)$ and $G(t,X)$ are continuous on $I\times Q$ ($I: 0\leqq t<\infty$, Q: an open set in R^n) and $\int_0^t\|G(s,X)\|\,ds$ is bounded for all t whenever X belongs to any compact subset of Q.

The following result of Yoshizawa [6] is well known.

Lemma 3.1. *Suppose that there exists a non-negative Liapunov function* $V(t,X)$ *on* $I\times Q$ *such that* $\dot{V}_{(3.1)}(t,X)\leqq -W(X)$, *where* $W(X)$ *is a positive definite with respect to a closed set* Ω *in the space* Q. *Moreover, suppose that* $F(t,X)$ *of the system* (3.1) *is bounded for all* t *when* X *belongs to an arbitrary compact set in* Q *and that* $F(t,X)$ *satisfies the following two conditions with respect to* Ω:

(a) $F(t,X)$ *tends to a function* $H(X)$ *for* $X\in\Omega$ *as* $t\to\infty$, *and on any compact set in* Ω *this convergence is uniform.*

(b) *Corresponding to each* $\varepsilon>0$ *and each* $Y\in\Omega$, *there exist a*

$\delta(\varepsilon, Y)$ and a $T(\varepsilon, Y)$ such that if $\|X-Y\|<\delta(\varepsilon, Y)$ and $t \geqq T(\varepsilon, Y)$, we have $\|F(t, X)-F(t, Y)\|<\varepsilon$.

Then, every bounded solution of (3.1) *approaches the largest semi-invariant set of the system $\dot{X}=H(X)$ contained in Ω as $t \to \infty$.*

Lemma 3.2. *Let $h(0)=0$, $xh(x)>0$ $(x \neq 0)$ and $\delta(t)-h'(x)>0$ $(\delta(t)>0)$, then*

$$2\delta(t)H(x) \geqq h^2(x) \qquad \left(H(x)=\int_0^x h(\xi)d\xi\right). \tag{3.2}$$

Proof of Lemma 3.2. We have $h^2(x)=2\int_0^x h'(\xi)h(\xi)d\xi$. Hence

$$\begin{aligned} 2\delta(t)H(x)-h^2(x) &= 2\int_0^x \{\delta(t)h(\xi)-h'(\xi)h(\xi)\}d\xi \\ &= 2\int_0^x \{\delta(t)-h'(\xi)\}h(\xi)d\xi \geqq 0. \qquad \text{Q.E.D.} \end{aligned}$$

4. Proof of Theorems. In the following, it will be assumed that $X=(x, y, z)$ and $\|X\|=\sqrt{x^2+y^2+z^2}$. CIP means the family of all continuous increasing positive definite functions and also CI means the family of all continuous increasing functions.

Proof of Theorem 1. Equation (1.2) is equivalent to the system

$$\dot{x}=y,\ \dot{y}=z,\ \dot{z}=-a(t)z-b(t)y-c(t)h(x)+e(t). \tag{4.1}$$

We denote $\gamma(t)=\int_0^t |c'(s)|\,ds$. It may be assumed that $\int_0^\infty |c'(t)|\,dt \leqq N<\infty$ and $\int_0^\infty |e(t)|\,dt \leqq E<\infty$. We define the Liapunov function $V(t, x, y, z)$ as

$$V(t, x, y, z)=e^{-\int_0^t |e(s)|ds}\{V_1(t, x, y, z)+k\}, \tag{4.2}$$

where

$$V_1(t, x, y, z)=e^{-\gamma(t)/c_0}V_0(t, x, y, z), \tag{4.3}$$

$$\begin{aligned} &V_0(t, x, y, z) \\ &= \mu c(t)H(x)+c(t)yh(x)+\frac{1}{2}b(t)y^2+\frac{1}{2}\mu a(t)y^2+\mu yz+\frac{1}{2}z^2 \end{aligned} \tag{4.4}$$

and k is a positive number to be determined later in the proof.

According to the conditions (i), (iii) and Lemma 3.2 it can be seen that

$$\begin{aligned} &V_0(t, z, y, z) \geqq \mu c_0\delta_1 H(x) \\ &\quad +\frac{1}{2}\left[\frac{1}{\rho}\left(\rho b_0-\frac{C}{1-\delta_1}\right)+\mu\{(a_0-\mu)-\mu\delta_2\}\right]y^2+\frac{\delta_2}{2(1+\delta_2)}z^2, \end{aligned} \tag{4.5}$$

where δ_1 and δ_2 are suitable positive numbers satisfying $1-\frac{C}{\rho b_0}>\delta_1>0$ and $\frac{a_0-\mu}{\mu}>\delta_2>0$. Then we can find a positive number δ_3 such that

$$V_0(t, x, y, z) \geqq \delta_3\{H(x)+y^2+z^2\}. \tag{4.6}$$

And it is easily verified that there exist two continuons functions $w_1(r)$ and $w_2(r)$ such that

(4.7) $\quad w_1(\|X\|)\leqq V(t,x,y,z)\leqq w_2(\|X\|)$ for all $X\in R^3$ and $t\in I$ where $w_1(r)\in CIP$, $w_1(r)\to\infty$ as $r\to\infty$ and $w_2(r)\in CI$.

It follows from (4.1), (4.3) and (4.4) that

$$\begin{aligned}\dot V_{0(4.1)}(t,x,y,z)=&-\{\mu b(t)-c(t)h'(x)\}y^2-\{a(t)-\mu\}z^2\\&+\frac{1}{2}\mu c'(t)\left\{2H(x)+\frac{2}{\rho h_1}yh(x)+\frac{1}{\rho^2 h_1}y^2\right\}\\&+\frac{1}{2}\left\{\mu a'(t)+b'(t)-\frac{1}{\rho}c'(t)\right\}y^2+(\mu y+z)e(t)\end{aligned}$$

and

$$\begin{aligned}&\dot V_{1(4.1)}(t,x,y,z)\\ \text{(4.8)}\qquad&\leqq-\frac{a_0b_0-Ch_1}{4}\cdot e^{-N/c_0}\left(y^2+\frac{2}{b_0}z^2\right)+e^{-r(t)/c_0}(\mu y+z)e(t).\end{aligned}$$

Let $k\geqq\dfrac{\mu^2+1}{4\delta_3}$, then by (4.1), (4.2), (4.6) and (4.8) we have

$$\text{(4.9)}\qquad \dot V_{(4.1)}(t,x,y,z)\leqq-\frac{a_0b_0-Ch_1}{4}\cdot e^{-(E+N/c_0)}\left(y^2+\frac{2}{b_0}z^2\right).$$

It follows from (4.7) and (4.9) that all solutions of (4.1) are uniform-bounded.

In the system (4.1) we set

$$\text{(4.10)}\qquad F(t,X)=\begin{pmatrix}y\\z\\-a(t)z-b(t)y-c(t)h(x)\end{pmatrix},\qquad G(t,X)=\begin{pmatrix}0\\0\\e(t)\end{pmatrix},$$

then $F(t,X)$ and $G(t,X)$ clearly satisfy the conditions of Lemma 3.1. Let $W(X)=\dfrac{a_0b_0-Ch_1}{4}\cdot e^{-(E+N/c_0)}\left(y^2+\dfrac{2}{b_0}z^2\right)$, then $\dot V_{(4.1)}(t,x,y,z)\leqq -W(X)$ and $W(X)$ is positive definite with respect to the closed set $\Omega\equiv\{(x,y,z)\mid x\in R^1, y=0, z=0\}$. It follows that on Ω

$$F(t,X)=\begin{pmatrix}0\\0\\-c(t)h(x)\end{pmatrix}.$$

According to the condition (v) and the boundedness of $c(t)$, we have $c(t)\to c_\infty$ as $t\to\infty$ where $0<c_0\leqq c_\infty\leqq C$. It is also clear that if we take

$$\text{(4.11)}\qquad H(X)=\begin{pmatrix}0\\0\\-c_\infty h(x)\end{pmatrix},$$

then conditions (a) and (b) of Lemma 3.1 are satisfied, and since all solutions of (4.1) are bounded, it follows from Lemma 3.1 that every solution of (4.1) approaches the largest semi-invariant set of $\dot X=H(X)$ contained in Ω as $t\to\infty$.

From (4.11), $\dot X=H(X)$ is the system $\dot x=0$, $\dot y=0$, $\dot z=-c_\infty h(x)$ which has the solution $x=c_1$, $y=c_2$, $z=c_3-c_\infty h(c_1)(t-t_0)$. To remain in Ω, $c_2=0$ and $c_3-c_\infty h(c_1)(t-t_0)=0$ for all $t\geqq t_0$ which implies $c_1=0$ and $c_3=0$.

Then the only solution of $\dot{X}=H(X)$ is $X\equiv 0$, i.e., the largest semi-invariant set of $\dot{X}=H(X)$ contained in Ω is the point $(0,0,0)$. Then it follows that $x(t)\to 0$, $y(t)\to 0$, $z(t)\to 0$ as $t\to\infty$. Q.E.D.

Proof of Corollary 1. In Theorem 1 we set $h(x)=x$ then the condition (ii) is satisfied. We take $h_1=h'(x)=1$, then $\mu=\rho$ and the conditions (iii)′ and (iv)′ are obtained. Thus we have the conclusion.

Q.E.D.

Proof of Theorem 2. Equation (1.3) is equivalent to the system

$$\dot{x}=y,\quad \dot{y}=z,\quad \dot{z}=-a(t)f(x,y)z-b(t)g(x,y)y-c(t)h(x)+e(t). \tag{4.12}$$

We denote $\alpha(t)=\int_0^t |a'(s)|\,ds$, $\beta(t)=\int_0^t |b'(s)|\,ds$ and $\gamma(t)=\int_0^t |c'(s)|ds$. It may be assumed that $\int_0^\infty |a'(t)|\,dt\leqq L<\infty$, $\int_0^\infty |b'(t)|\,dt\leqq M<\infty$, $\int_0^\infty |c'(t)|dt \leqq N<\infty$ and $\int_0^\infty |e(t)|\,dt\leqq E<\infty$. We define the Liapunov function $V(t,x,y,z)$ as

$$V(t,x,y,z)=e^{-\int_0^t |e(s)|ds}\{V_1(t,x,y,z)+k\}, \tag{4.13}$$

where

$$V_1(t,x,y,z)=e^{-\{\alpha(t)/a_0+\beta(t)/b_0+\gamma(t)/c_0\}}\cdot V_0(t,x,y,z)\equiv e^{-R(t)}V_0(t,x,y,z), \tag{4.14}$$

$$\begin{aligned}V_0(t,x,y,z)=&\,\mu c(t)H(x)+c(t)yh(x)+b(t)\int_0^y g(x,\eta)\eta d\eta\\ &+\mu a(t)\int_0^y f(x,\eta)\eta d\eta+\mu yz+\frac{1}{2}z^2\end{aligned} \tag{4.15}$$

and k is a positive constant to be determined later in the proof.

Since $\mu\dfrac{b(t)}{c(t)}g_0\geqq h'(s)$, we can use Lemma 3.2 and we have

$$\left\{\mu H(x)+yh(x)+\frac{1}{2}\frac{b(t)}{c(t)}g_0y^2\right\}\geqq\left\{\sqrt{\mu H(x)}-|y|\sqrt{\frac{b_0g_0}{2C}}\right\}^2\geqq 0. \tag{4.16}$$

Then we can find a positive number δ_1 such that

$$V_0(t,x,y,z)\geqq\delta_1\{H(x)+y^2+z^2\}. \tag{4.17}$$

And it is easily verified that there exist two continuous functions $w_1(r)$ and $w_2(r)$ such that

$$w_1(\|X\|)\leqq V(t,x,y,z)\leqq w_2(\|X\|)\qquad\text{for all } X\in R^3 \text{ and } t\in I \tag{4.18}$$

where $w_1(r)\in CIP$, $w_1(r)\to\infty$ as $r\to\infty$ and $w_2(r)\in CI$.

Along any solution $(x(t), y(t), z(t))$ of (4.12) we have

$$\begin{aligned}&\dot{V}_{0(4.12)}(t,x,y,z)\\ &\leqq-\{\mu b(t)g(x,y)-c(t)h'(x)\}y^2-\{a(t)f(x,y)-\mu\}z^2+(\mu y+z)e(t)\\ &\quad+c'(t)\left\{\mu H(x)+yh(x)+\frac{1}{2}\frac{b(t)}{c(t)}g_0y^2\right\}-\frac{1}{2}c'(t)\frac{b(t)}{c(t)}g_0y^2\\ &\quad+b'(t)\int_0^y g(x,\eta)\eta d\eta+\mu a'(t)\int_0^y f(x,\eta)\eta d\eta\end{aligned}$$

and

$$\dot V_{1(4.12)}(t, x, y, z) \tag{4.19}$$
$$\leqq -\frac{a_0b_0f_0g_0-Ch_1}{4}e^{-Q}\left(y^2+\frac{2}{b_0g_0}z^2\right)+e^{-R(t)}(\mu y+z)e(t)$$

where $Q=\frac{L}{a_0}+\frac{M}{b_0}+\frac{N}{c_0}$.

Let $k\geqq\frac{\mu^2+1}{4\delta_1}$, then it follows from (4.12), (4.13), (4.17) and (4.19) that

$$\dot V_{(4.12)}(t, x, y, z)\leqq -\frac{a_0b_0f_0g_0-Ch_1}{4}e^{-(E+Q)}\left(y^2+\frac{2}{b_0g_0}z^2\right). \tag{4.20}$$

The remainder of the proof now proceeds as in that of Theorem 1.

Q.E.D.

Remark. *It turns out from the proofs that above theorems can be extended to the following cases*

$$\dddot x+a(t)\ddot x+b(t)\dot x+c(t)x=e(t, x, \dot x, \ddot x), \tag{1.1)'}$$

$$\dddot x+a(t)\ddot x+b(t)\dot x+c(t)h(x)=e(t, x, \dot x, \ddot x), \tag{1.2)'}$$

$$\dddot x+a(t)f(x, \dot x)\ddot x+b(t)g(x, \dot x)\dot x+c(t)h(x)=e(t, x, \dot x, \ddot x), \tag{1.3)'}$$

where $e(t, x, y, z)$ *is continuous in* $I\times R^3$. *The condition* (vi) *of Theorem* 1 (*condition* (viii) *of Theorem* 2) *is then modified as*

(vi)′ ((viii)′) $|e(t, x, y, z)|\leqq\tilde e(t)$ *for all* $(x, y, z)\in R^3$ *where* $\tilde e(t)$ *is continuous in* I *and satisfies* $\int_0^\infty \tilde e(t)dt<\infty$.

The proofs run just as before, using $\tilde e(s)$ in place of $e(s)$ in (4.2) and (4.13) e.g.

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