201. *On the Stability of Solutions of Certain Third Order Ordinary Differential Equations*

By Tadayuki HARA
Osaka University

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1. Introduction. In this note we consider the asymptotic stability in the large of each of the zero solutions of the differential equations

(1.1) $$\dddot{x}+a(t)\ddot{x}+b(t)\dot{x}+c(t)x=0 \qquad \left(\dot{x}=\frac{dx}{dt}\right),$$

(1.2) $$\dddot{x}+a(t)f(x,\dot{x})\ddot{x}+b(t)g(x,\dot{x})\dot{x}+c(t)x=0,$$

where $a(t)$, $b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty)$ and $f(x,y)$, $f_x(x,y)$, $g(x,y)$ and $g_x(x,y)$ are continuous and real valued for all (x,y).

The zero solution of (1.1)(or (1.2)) is called asymptotically stable in the large, if it is stable and if every solution of (1.1) (or (1.2)) tends to zero as $t\to\infty$.

Many results have been obtained on the asymptotic property of autonomous equations of third order (cf. [1]).

In [2], K. E. Swick established conditions under which all the solutions of non-autonomous equations

(1.3) $$\dddot{x}+p(t)\ddot{x}+q(t)g(\dot{x})+r(t)h(x)=0,$$

(1.4) $$\dddot{x}+f(t,x,\dot{x})\ddot{x}+q(t)g(\dot{x})+r(t)h(x)=0,$$

tend to the zero solution as $t\to\infty$. We consider somewhat different non-autonomous equations (1.1) and (1.2) in which $a(t)$, $b(t)$ and $c(t)$ may oscillate to some extent.

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2. Theorems.

Theorem 1. *Suppose that $a(t)$, $b(t)$ and $c(t)$ are continuously differentiable on $[0,\infty)$ and following conditions are satisfied;*

(i) $A\geqq a(t)\geqq a_0>0$, $B\geqq b(t)\geqq b_0>0$, $C\geqq c(t)\geqq c_0>0$ *for* $t\in I=[0,\infty)$,

(ii) $a_0b_0-C>0$,

(iii) $\mu a'(t)+b'(t)-\dfrac{1}{\mu}c'(t)<\dfrac{a_0b_0-C}{2} \qquad \left(\mu=\dfrac{a_0b_0+C}{2b_0}\right)$,

(iv) $\int_0^\infty |c'(t)|\,dt<\infty$, $c'(t)\to 0$ as $t\to\infty$.

Then every solution $x(t)$ of (1.1) *is uniform-bounded and satisfies* $x(t)\to 0$, $\dot{x}(t)\to 0$, $\ddot{x}(t)\to 0$ *as* $t\to\infty$.

Remark. If (1.1) is the differential equation with constant coefficients $\dddot{x}+a\ddot{x}+b\dot{x}+cx=0$, then the conditions above reduce to the Routh-Hurwitz conditions $a>0$, $c>0$ and $ab-c>0$. It follows from conditions (iii) and (iv) that $a(t)$ and $b(t)$ may be periodic functions and $c(t)$ may approach to some constant with a damped oscillation. If $c(t)$ is a bounded monotone function, the condition (iv) is satisfied.

Theorem 2. *Suppose that* $a(t)$, $b(t)$ *and* $c(t)$ *are continuously differentiable on* $[0,\infty)$ *and following conditions are satisfied;*

(i) $A\geqq a(t)\geqq a_0>0,\ B\geqq b(t)\geqq b_0>0,\ C\geqq c(t)\geqq c_0>0$ *for* $t\in I=[0,\infty)$,

(ii) $f(x,y)\geqq f_0>0,\ yf_x(x,y)\leqq 0$ *for all* (x,y),

(iii) $g(x,y)\geqq g_0>0,\ yg_x(x,y)\leqq 0$ *for all* (x,y),

(iv) $a_0b_0f_0g_0-C>0$,

(v) $\dfrac{\mu^2}{a_0}|a'(t)|+\dfrac{|b'(t)|\,c(t)}{\mu b_0}-\dfrac{c'(t)}{\mu}<\dfrac{a_0b_0f_0g_0-C}{2}\qquad\left(\mu=\dfrac{a_0b_0f_0g_0+C}{2b_0g_0}\right)$,

(vi) $\int_0^\infty|a'(t)|\,dt<\infty,\ \int_0^\infty|b'(t)|\,dt<\infty,\ \int_0^\infty|c'(t)|\,dt<\infty,\ c'(t)\to 0$ *as* $t\to\infty$.

Then every solution $x(t)$ *of* (1.2) *is uniform-bounded and satisfies* $x(t)\to 0,\ \dot{x}(t)\to 0,\ \ddot{x}(t)\to 0$ *as* $t\to\infty$.

3. Auxiliary Lemmas.

Consider the system

(3.1) $\dot{X}=F(t,X),\ F(t,0)\equiv 0$ for $t\in I=[0,\infty),\ F(t,X)\in C^0(I\times R^n)$.

The following results are well known (cf. [3]).

Lemma 3.1. *Suppose that there exists a Liapunov function* $V(t,X)$ *defined on* $0\leqq t<\infty,\ \|X\|<H\ (H>0)$ *which satisfies the following conditions;*

(i) $V(t,0)\equiv 0$,

(ii) $a(\|X\|)\leqq V(t,X)$, *where* $a(r)\in CIP$ *(i.e. continuous and increasing positive definite functions),*

(iii) $\dot{V}_{(3.1)}(t,X)\leqq 0$.

Then the solution $x(t)\equiv 0$ *of the system* (3.1) *is stable.*

Lemma 3.2. *Suppose that there exists a Liapunov function* $V(t,X)$ *defined on* $0\leqq t<\infty,\ \|X\|\geqq R$, *where* R *may be large, which satisfies the following conditions;*

(i) $a(\|X\|)\leqq V(t,X)\leqq b(\|X\|)$, *where* $a(r)\in CI$ *(i.e. continuous increasing functions),* $a(r)\to\infty$ *as* $r\to\infty$ *and* $b(r)\in CI$.

(ii) $\dot{V}_{(3.1)}(t,X)\leqq 0$.

Then the solutions of (3.1) *are uniform-bounded.*

Lemma 3.3. *Suppose that there exists a non-negative Liapunov function* $V(t,X)$ *defined on* $I\times R^n$ *such that* $\dot{V}_{(3.1)}(t,X)\leqq -W(X)$, *where* $W(X)$ *is positive definite with respect to a closed set* Ω *in the space* R^n. *Moreover, suppose that* $F(t,X)$ *of the system* (3.1) *is bounded for all* t *when* X *belongs to an arbitrary compact set in* R^n *and that there is a* *function $H(X)$ defined on Ω such that;*

(a) *$F(t, X)$ tends to $H(X)$ for $X \in \Omega$ as $t \to \infty$ and on any compact set in Ω this convergence is uniform.*

(b) *Corresponding to each $\varepsilon > 0$ and each $Y \in \Omega$, there exist a $\delta(\varepsilon, Y) > 0$ and a $T(\varepsilon, Y)$ such that if $\|x - y\| < \delta(\varepsilon, Y)$ and $t \geqq T(\varepsilon, Y)$, we have $\|F(t, X) - F(t, Y)\| < \varepsilon$.*

Then, every bounded solution of (3.1) *approaches the largest semi-invariant set of the system $\dot{X} = H(X)$ contained in Ω as $t \to \infty$. In particular, if all solutions of* (3.1) *are bounded, every solution of* (3.1) *approaches the largest semi-invariant set of $\dot{X} = H(X)$ contained in Ω as $t \to \infty$.*

4. Proof of Theorems.

In this section it will be assumed that $X = (x, y, z)$ and $\|X\| = \sqrt{x^2 + y^2 + z^2}$.

Proof of Theorem 1. We consider, in place of (1.1), the equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -a(t)z - b(t)y - c(t)x, \tag{4.1}$$

and denote $\gamma(t) = \int_0^t |c'(s)|\, ds$. It may be assumed that $\int_0^\infty |c'(t)|\, dt \leqq N < \infty$. We define the Liapunov function $V(t, x, y, z)$ as

$$V(t, x, y, z) = e^{-\gamma(t)/c_0} V_0(t, x, y, z), \tag{4.2}$$

where

$$V_0(t, x, y, z) = \frac{1}{2}\mu c(t) x^2 + c(t) xy + \frac{1}{2}[b(t) + \mu a(t)] y^2 + \mu yz + \frac{1}{2} z^2. \tag{4.3}$$

An easy calculation shows that

$$\begin{aligned} &\frac{1}{2}\left[\mu c_0 \left(x + \frac{y}{\mu}\right)^2 + \frac{1}{\mu}\{(\mu b_0 - C) + \mu^2(a_0 - \mu)\} y^2 + (z + \mu y)^2\right] \\ &\leqq V_0(t, x, y, z) \\ &\leqq \frac{1}{2}\left[\mu C \left(x + \frac{y}{\mu}\right)^2 + \frac{1}{\mu} \cdot \{(\mu B - c_0) + \mu^2(A - \mu)\} y^2 + (z + \mu y)^2\right]. \end{aligned} \tag{4.4}$$

According to the condition (ii), we obtain $(\mu b_0 - C) + \mu^2(a_0 - \mu) > 0$ and $(\mu B - c_0) + \mu^2(A - \mu) > 0$, thus it is easily verified that there exist positive numbers δ_1 and δ_2 such that

$$\delta_1(x^2 + y^2 + z^2) \leqq \frac{1}{2}\left[\mu c_0 \left(x + \frac{y}{\mu}\right)^2 + \frac{1}{\mu}\{(\mu b_0 - C) + \mu^2(a_0 - \mu)\} y^2 + (z + \mu y)^2\right]$$

and

$$\frac{1}{2}\left[\mu C \left(x + \frac{y}{\mu}\right)^2 + \frac{1}{\mu}\{(\mu B - c_0) + \mu^2(A - \mu)\} y^2 + (z + \mu y)^2\right] \leqq \delta_2(x^2 + y^2 + z^2).$$

Then we have $\delta_1(x^2 + y^2 + z^2) \leqq V_0(t, x, y, z) \leqq \delta_2(x^2 + y^2 + z^2)$ and the following inequality is obtained on referring to the relation (4.2),

$$\delta_1 e^{-N/c_0}(x^2 + y^2 + z^2) \leqq V(t, x, y, z) \leqq \delta_2(x^2 + y^2 + z^2). \tag{4.5}$$

It follows from (4.1) and (4.3) that

$$\dot{V}_{0(4.1)}(t, x, y, z) = -[\mu b(t) - c(t)]y^2 - [a(t) - \mu]z^2 + \frac{1}{2}\mu c'(t)\left(x + \frac{y}{\mu}\right)^2 + \frac{1}{2}\left[\mu a'(t) + b'(t) - \frac{1}{\mu}c'(t)\right]y^2.$$

By (4.1) and (4.2) we have

$$\dot{V}_{(4.1)}(t, x, y, z) = e^{-\gamma(t)/c_0}\left\{\dot{V}_{0(4.1)}(t, x, y, z) - \frac{|c'(t)|}{c_0}V_0(t, x, y, z)\right\}.$$

Using the inequality (4.4) and the fact that $c'(t) - |c'(t)| \leqq 0$, we have

$$\begin{aligned}
&\left\{\dot{V}_{0(4.1)}(t, x, y, z) - \frac{|c'(t)|}{c_0} \cdot V_0(t, x, y, z)\right\} \\
&\leqq -[\mu b(t) - c(t)]y^2 - [a(t) - \mu]z^2 + \frac{1}{2}\mu c'(t)\left(x + \frac{y}{\mu}\right)^2 \\
&\quad + \frac{1}{2}\left[\mu a'(t) + b'(t) - \frac{1}{\mu}c'(t)\right]y^2 \\
&\quad - \frac{1}{2} \cdot \frac{|c'(t)|}{c_0} \cdot \left[\mu c_0\left(x + \frac{y}{\mu}\right)^2 + \frac{1}{\mu}\{(\mu b_0 - C) + \mu^2(a_0 - \mu)\}y^2 + (z + \mu y)^2\right] \\
&\leqq -[\mu b(t) - c(t)]y^2 - [a(t) - \mu]z^2 + \frac{1}{2}\left[\mu a'(t) + b'(t) - \frac{1}{\mu}c'(t)\right]y^2.
\end{aligned}$$

According to the conditions (i), (ii), (iii) and above inequality, we have

$$\dot{V}_{(4.1)}(t, x, y, z) \leqq -\frac{a_0 b_0 - C}{4} \cdot e^{-N/c_0}\left(y^2 + \frac{2}{b_0}z^2\right). \tag{4.6}$$

It now follows from (4.5), (4.6), Lemma 3.1 and Lemma 3.2 that the zero solution of (4.1) is stable and that all solutions of (4.1) are uniform-bounded.

In the following, Lemma 3.3 plays the important role to complete the proof. In the system (4.1) we set

$$F(t, X) = \begin{pmatrix} y \\ z \\ -a(t)z - b(t)y - c(t)x \end{pmatrix}.$$

Let $W(X) = \frac{a_0 b_0 - C}{4} \cdot e^{-N/c_0}\left(y^2 + \frac{2}{b_0}z^2\right)$, then $\dot{V}_{(4.1)}(t, x, y, z) \leqq -W(X)$ and $W(X)$ is positive definite with respect to the closed set $\Omega \equiv \{(x, y, z) \mid x \in R^1, y = 0, z = 0\}$. Since $a(t)$, $b(t)$ and $c(t)$ are bounded for all $t \in I$, $F(t, X)$ is bounded for all $t \in I$ when X belongs to an arbitrary compact set in R^3. It follows that on Ω

$$F(t, X) = \begin{pmatrix} 0 \\ 0 \\ -c(t)x \end{pmatrix}.$$

According to the condition (iv) and the boundedness of $c(t)$, we have $c(t) \to c_\infty$ as $t \to \infty$ where $0 < c_0 \leqq c_\infty \leqq C$. It is also clear that if we take

$$H(X)=\begin{pmatrix} 0 \\ 0 \\ -c_\infty X \end{pmatrix}, \tag{4.7}$$

then conditions (a) and (b) of Lemma 3.3 are satisfied, and since all solutions of (4.1) are bounded, it follows from Lemma 3.3 that every solution of (4.1) approaches the largest semi-invariant set of $\dot{X}=H(X)$ contained in Ω as $t\to\infty$.

From (4.7), $\dot{X}=H(X)$ is the system $\dot{x}=0$, $\dot{y}=0$, $\dot{z}=-c_\infty x$ which has the solution $x=c_1$, $y=c_2$, $z=c_3-c_\infty c_1(t-t_0)$.

To remain in Ω, $c_2=0$ and $c_3-c_\infty c_1(t-t_0)=0$ for all $t\geqq t_0$ which implies $c_1=0$ and $c_3=0$. Then the only solution of $\dot{X}=H(X)$ is $X\equiv 0$, i.e., the largest semi-invariant set of $\dot{X}=H(X)$ contained in Ω is the set $\{(0,0,0)\}$. Thus it follows that $x(t)\to 0$, $y(t)\to 0$, $z(t)\to 0$ as $t\to\infty$.

Proof of Theorem 2. Equation (1.2) is equivalent to the system

$$\dot{x}=y,\ \dot{y}=z,\ \dot{z}=-a(t)f(x,y)z-b(t)g(x,y)y-c(t)x. \tag{4.8}$$

We denote $\alpha(t)=\int_0^t|a'(s)|\,ds$, $\beta(t)=\int_0^t|b'(s)|\,ds$, and $\gamma(t)=\int_0^t|c'(s)|\,ds$. It may be assumed that $\int_0^\infty|a'(t)|\,dt\leqq L<\infty$, $\int_0^\infty|b'(t)|\,ds\leqq M<\infty$ and $\int_0^\infty|c'(t)|\,dt\leqq N<\infty$. We define the Liapunov function $V(t,x,y,z)$ as

$$V(t,x,y,z)=e^{-[\alpha(t)/a_0+\beta(t)/b_0+\gamma(t)/c_0]}V_0(t,x,y,z), \tag{4.9}$$

where

$$V_0(t,x,y,z)=\frac{1}{2}\mu c(t)x^2+c(t)xy+b(t)\int_0^y g(x,\eta)\eta d\eta + \mu a(t)\int_0^y f(x,\eta)\eta d\eta+\mu yz+\frac{1}{2}z^2. \tag{4.10}$$

We may also write $V_0(t,x,y,z)$ as follows;

$$\begin{aligned} V_0(t,x,y,z) &= \frac{1}{2}\mu c(t)\left(x+\frac{y}{\mu}\right)^2-\frac{c(t)}{2\mu}y^2+b(t)\int_0^y g(x,\eta)\eta d\eta+\frac{1}{2}(z+\mu y)^2 \\ &\quad -\frac{1}{2}\mu^2y^2+\mu a(t)\int_0^y f(x,\eta)\eta d\eta \\ &= \frac{1}{2}\mu c(t)\cdot\left(x+\frac{y}{\mu}\right)^2+\frac{1}{\mu}\int_0^y\{\mu b(t)\cdot g(x,\eta)-c(t)\}\eta d\eta \\ &\quad +\frac{1}{2}(z+\mu y)^2+\mu\int_0^y\{a(t)f(x,\eta)-\mu\}\eta d\eta. \end{aligned} \tag{4.11}$$

Then it follows that

$$\frac{1}{2}e^{-(L/a_0+M/b_0+N/c_0)}\left[\mu c_0\left(x+\frac{y}{\mu}\right)^2+\frac{1}{\mu}\{(\mu b_0 g_0-C) + \mu^2(a_0 f_0-\mu)\}y^2+(z+\mu y)^2\right]$$

$$(4.12)\quad \leqq V(t,x,y,z)$$
$$\leqq \frac{1}{2}\mu C\left(x+\frac{y}{\mu}\right)^2+\frac{1}{\mu}\int_0^y [\{\mu Bg(x,\eta)-c_0\}+\mu^2\{Af(x,\eta)-\eta\}]\eta d\eta$$
$$+\frac{1}{2}(z+\mu y)^2.$$

According to the condition (iv), we obtain $(\mu b_0 g_0 - C)+\mu^2(a_0 f_0-\mu)>0$ and $\{\mu Bg(x,\eta)-c_0\}+\{Af(x,\eta)-\eta\}>0$, thus it is easily verified that the left-hand side of (4.12) is positive definite and the right-hand side is a positive continuous function.

Along any solution $(x(t), y(t), z(t))$ of (4.8) we have

$$\dot V_{0(4.8)}(t,x,y,z) = -[\mu b(t)g(x,y)-c(t)]y^2-[a(t)f(x,y)-\mu]z^2$$
$$+b(t)y\int_0^y g_x(x,\eta)\eta d\eta+\mu a(t)y\int_0^y f_x(x,\eta)\eta d\eta+\frac{1}{2}\mu c'(t)\left(x+\frac{y}{\mu}\right)^2$$
$$-\frac{c'(t)}{2\mu}y^2+b'(t)\int_0^y g(x,\eta)\eta d\eta+\mu a'(t)\int_0^y f(x,\eta)\eta d\eta.$$

By (4.8) and (4.9) we have

$$\dot V_{(4.8)}(t,x,y,z)$$
$$=e^{-[\alpha(t)/a_0+\beta(t)/b_0+\gamma(t)/c_0]}\left\{\dot V_{0(4.8)}-\left(\frac{|a'(t)|}{a_0}+\frac{|b'(t)|}{b_0}+\frac{|c'(t)|}{c_0}\right)V_0\right\}.$$

The following calculation is proceeded in a manner similar to that of Theorem 1.

$$\left\{\dot V_{0(4.8)}(t,x,y,z)-\left(\frac{|a'(t)|}{a_0}+\frac{|b'(t)|}{b_0}+\frac{|c'(t)|}{c_0}\right)V_0(t,x,y,z)\right\}$$
$$\leqq -[\mu b(t)g(x,y)-c(t)]y^2-[a(t)f(x,y)-\mu]z^2+b(t)y\int_0^y g_x(x,\eta)\eta d\eta$$
$$+\mu a(t)y\int_0^y f_x(x,\eta)\eta d\eta+\frac{1}{2}\left\{\frac{\mu^2}{a_0}|a'(t)|+\frac{|b'(t)|\,c(t)}{\mu b_0}-\frac{c'(t)}{\mu}\right\}y^2.$$

According to the conditions (i)$\sim$(v) and the above inequality, we have

$$(4.13)\qquad \dot V_{(4.8)}(t,x,y,z)\leqq -\frac{a_0b_0f_0g_0-C}{4}e^{-(L/a_0+M/b_0+N/c_0)}\left(y^2+\frac{2}{b_0g_0}z^2\right).$$

The remainder of the proof now proceeds as in Theorem 1.

Q.E.D.

References

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