127. *The Order of Fourier Coefficients of Function of Higher Variation*

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This paper contains two theorems. First we estimate the order of Fourier coefficients of function of Wiener's class V_p which is strictly larger class than that of the class of functions of bounded variation. We have been able to find out the best constant which terms out to be $V_p(f)\pi^{-1}2^{1/q}$ in our case. The second theorem concerns about how many Fourier coefficients can have exactly the order $n^{-1/p}$.

1. Let f be a real valued 2π-periodic function defined on $[0, 2\pi]$ and let $0=t_0\le t_1\le t_2\le \cdots \le t_n=2\pi$ be a partition of $[0, 2\pi]$. We write, for $1\le p<\infty$,

$$V_p(f)=\sup\left\{\sum_{i=1}^{n}|f(t_i)-f(t_{i-1})|^p\right\}^{1/p} \tag{1}$$

where sup is taken over all partitions of $[0, 2\pi]$. We say that a function f belongs to V_p or f is the function of p-th variation if $V_p(f)<\infty$. In terms of Wiener [5] we denote the class of all 2π-periodic functions of p-th variation on the segment $[0, 2\pi]$ by V_p. We call $V_p(f)$ the p-th total variation of f. It can easily be verified that

$$V_p\subset V_q \qquad (1\le p<q<\infty) \tag{2}$$

is a strict inclusion. For $p=1$, V_1 is the class of functions of bounded variation. Let

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{3}$$

be a Fourier series of f. In the case V_1 the following theorem is well known [1] (see also [7]).

Theorem A. *If f belongs to V_1 then*

$$|a_n|\le V_1(f)(\pi n)^{-1}; \qquad |b_n|\le V_1(f)(\pi n)^{-1} \tag{4}$$

for all $n>1$, where $V_1(f)$ is the first total variation of f over $[0, 2\pi]$.

Recently M. Taibleson [3] has proved a weaker form of Theorem A by an elementary method (see also [1] page 210). M. and S. Izumi [2] have given another elementary proof of Theorem A with the best constant $V_1(f)\pi^{-1}$ in (4). We extend Theorem A in the following way.

Theorem 1. *If f belongs to V_p $(1\le p<\infty)$ then*

$$\begin{cases}|a_n|\le V_p(f)\pi^{-1}2^{1/q}n^{-1/p}; \\ |b_n|\le V_p(f)\pi^{-1}2^{1/q}n^{-1/p}\end{cases} \tag{5}$$

for all $n>1$, *where* $V_p(f)$ *denotes the p-th total variation of* f *over* $[0,2\pi]$ *and* $1/p+1/q=1$.

Proof of Theorem 1. Since

$$
\begin{aligned}
\pi a_n &= \int_0^{2\pi} f(x)\cos nx\,dx = \int_{-\pi/2n}^{2\pi-\pi/2n} f(x)\cos nx\,dx \\
&= \sum_{k=0}^{2n-1}(-1)^k \int_{-\pi/2n}^{\pi/2n} f\left(x+\frac{k\pi}{n}\right)\cos nx\,dx \\
&= \int_{-\pi/2n}^{\pi/2n}\left[\sum_{j=0}^{n=1}\left\{f\left(x+\frac{2j\pi}{n}\right)-f(x+(2j+1)\pi/n)\right\}\right]\cos nx\,dx \\
&= -\int_{-\pi/2n}^{\pi/2n}\left[\sum_{j=0}^{n-1}\{f(x+(2j+1)\pi/n)-f(x+(2j+2)\pi/n)\}\right]\cos nx\,dx
\end{aligned}
\tag{6}
$$

and hence we can write

$$
2\pi|a_n| \le \int_{-\pi/2n}^{\pi/2n}\left[\sum_{k=0}^{2n-1}\left|f\left(x+\frac{k\pi}{n}\right)-f(x+(k+1)\pi/n)\right|\right]\cos nx\,dx. \tag{7}
$$

But $\cos nx \ge 0$ in $[-\pi/2n, \pi/2n]$, applying Hölder's inequality on the integrand of (7) we get,

$$
2\pi|a_n| \le \int_{-\pi/2n}^{\pi/2n}\left[\left\{\sum_{k=0}^{2n-1}\left|f\left(x+\frac{k\pi}{n}\right)-f(x+(k+1)\pi/n)\right|^p\right\}^{1/p} \times\left\{\sum_{k=0}^{2n-1}1^q\right\}^{1/q}\right]\cos nx\,dx
$$

for $1/p+1/q=1$. But using (1) above we get

$$
2\pi|a_n| \le V_p(f)(2n)^{1/q}\int_{-\pi/2n}^{\pi/2n}\cos nx\,dx = V_p(f)(2n)^{1/q}2/n.
$$

This gives the first inequality of (5). The second is also similarly proved.

Remark 1. Since in the case $p=1$ in our Theorem 1, q becomes infinity and hence the constant $V_p(f)\pi^{-1}2^{1/q}$ reduces to $V_1(f)\pi^{-1}$ which is the best constant in Theorem A. Hence $V_p(f)\pi^{-1}2^{1/q}$ is the best constant in our Theorem 1.

2. Now we study how many Fourier coefficients can have exactly the order $n^{-1/p}$ in the following;

Theorem 2. *If* f *belongs to* $V_p(1<p<\infty)$ *and* $\{n_k\}$ *denotes the sequence of* n *such that* $\rho_n > A/n^{1/p}$ *where* A *is a fixed constant and* $\rho_n = \sqrt{a_n^2+b_n^2}$ *then*

$$
\sum_{k=1}^{N} n_k = 0(n_N) \qquad (N\to\infty).
$$

For the proof of above theorem we shall need the following lemma which is due to Young [6].

Lemma (Young). *If* f *belongs to* V_p $(1\le p<\infty)$ *then*

$$
\omega_p(\delta, f) = \sup_{|h|\le\delta}\left\{\int_0^{2\pi}|f(t+h)-f(t)|^p\,dt\right\}^{1/p} \le \delta^{1/p}V_p(f).
$$

Proof of Theorem 2. *Case* 1. Suppose $1<p<2$. Then from hypothesis and Theorem 1, we can conclude

$$
c_1 n_k^{-1/p} \le \rho_{n_k}(f) \le c_2 n_k^{-1/p} \qquad (k=1,2,\cdots). \tag{8}
$$

We can write from (3),

$$(9) \qquad f(t+h)-f(t-h)\sim 2\sum_{k=1}^{\infty}(b_k \cos kt - a_k \sin kt)\sin kh.$$

Since $1/p+1/q=1$, using the Hausdorff inequality, (9) gives

$$\left(\sum_{k=1}^{\infty}\rho_k^q |\sin kh|^q\right)^{1/q} \le c_p \|f(t+h)-f(t-h)\|_p \le c_p \omega_p(|2h|, f)$$

where c_p is a constant depending only on p. Therefore

$$n^{-q}\sum_{k=1}^{n} k^q \rho_k^q = 0\left(\omega_p^q\left(\frac{\pi}{n}, f\right)\right).$$

Using (9) and lemma of Young, we get

$$\sum_{k=1}^{N} n_k^{(1-1/p)q} = 0(n_N^{(1-1/p)q}).$$

That is

$$\sum_{k=1}^{N} n_k = 0(n_N) \qquad (N\to\infty).$$

Case 2. Suppose $2\le p<\infty$. Using Parseval equation on (9) we get

$$\left(4\pi\sum_{k=1}^{\infty}\rho_k^2 \sin^2 kh\right)^{1/2} = \|f(t+h)-f(t-h)\|_2$$
$$\le c_p \|f(t+h)-f(t-h)\|_p = 0(\omega_p(|2h|, f)) \qquad (h\to +0).$$

Hence we get

$$n^{-2}\sum_{k=1}^{n} k^2\rho_k^2 = 0\left(\omega_p^2\left(\frac{\pi}{n}\right), f\right)\right) \qquad (n\to\infty).$$

Now using lemma of Young and by given hypothesis we get

$$(10) \qquad \sum_{k=1}^{N} n_k^{2(1-1/p)} = 0(n_N^{2(1-1/p)}) \qquad (N\to\infty).$$

S. B. Steckin [4] has shown that the above condition (10) implies

$$\sum_{k=1}^{N} n_k = 0(n_N) \qquad (N\to\infty).$$

Hence Theorem 2 is completely proved.

Remark 2. The above theorem is not true for $p=1$ (for the class of functions of bounded variation). For the function

$$f(t)=\sum_{k=1}^{\infty}\frac{\sin kt}{k}$$

belongs to the class V_1 but the condition

$$\sum_{k=1}^{N} n_k = 0(n_N) \qquad (N\to\infty)$$

does not satisfy for $n_k=k$ $(k=1,2,3,\cdots)$.

From Theorem 2 we can also deduce the following;

Corollary. *The sequence* $\{n_k\}$ *in Theorem* 2 *can be split into a finite number of lacunary subsequences* (see [4] page 388).

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