91. *Note on Topological Transitivity*

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M. Morse and G. A. Hedlund solved the problem of the topological transitivity for each two-dimensional closed orientable Riemannian manifold $\sum$ of class C^3 and of genus $p>1$ provided that no geodesic on $\sum$ has on it two mutually conjugate points [4]. I have shown the one method of symbolic representation already [1]. In this paper we shall show the new proof of topological transitivity as an application of the symbolic representation. (Cf. Morse-Hedlund [2],[3].)

1. We already know the following theorems of symbolic representation.

Theorem 1. If there be given any regular geodesic relative to P on $\sum$, there exists one, and only one unending regular sequence whose generating symbols are $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{a}_i^{-1}$, $\tilde{b}_i^{-1}$.

Theorem 2. If there be given any unending regular sequence whose generating symbols are $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{a}_i^{-1}$, $\tilde{b}_i^{-1}$, there exists at least one geodesic which corresponds to the given regular sequence.

Now we prepare some definitions.

Definition 1. Any geodesic or geodesic ray on $\sum$ is represented by a curve on phase space Ω of $\sum$. If its closure coincides with Ω, we say that the geodesic or geodesic ray is transitive.

Definition 2. Any symbolic ray will be termed transitive if it contains a copy of all regular subblocks.

2. Lemma 1. There exists a transitive regular symbolic ray.

Proof. As the set of regular blocks is enumerable, we denote them $A_1, A_2, A_3, \ldots$.
Then the ray

$$X = A_1e_1\, A_2e_2\, A_3e_3 \ldots$$

is regular if the symbols e_i are successively chosen so as to satisfy the conditions (1) and (2) of regular sequence. (Cf. [1].) It is evident that X is transitive.

Theorem 3. In the case $p>1$ if the non-conjugacy hypothesis holds good, two geodesic rays with the same initial point on Φ can not be of the same type.

Proof. Let two geodesic rays r_1, r_2 with the same initial point on Φ be of the same type and f be the mapping explained in my paper [1]. Morse-Hedlund showed that $f^{-1}(r_1)$ and $f^{-1}(r_2)$ are not of the same type. Then r_1 and r_2 are not of the same type. This contradicts the assumption.

Lemma 2. Suppose that there is no geodesic ray which is of the same type with the given geodesic ray r_1 passing the point P of Φ. Then, corresponding to two arbitrary positive constants ε, d there exists a positive constant h so large that if each point of the geodesic segment G_1 of r_1 with a common initial point, of length h, lies within a geodesic distance d of some point of a second geodesic segment G_2 which has a common initial point with G_1, then G_2 has at least one element within ε of the element of G_1 which lies at the midpoint of G_1.

Proof. Suppose that this Lemma is not true. There exists a positive constant h so large that even if each point of G_1, of length h, lies within a geodesic distance d of some point of G_2, every element of G_2 lies without ε of the element of G_1 which lies at the midpoint of G_1. When ε is fixed and h tends to infinite, it will be seen that G_1 and G_2 are of the same type and this is contrary to the assumption.

Lemma 3. Suppose that there is no geodesic ray which is of the same type with the given geodesic ray r_1 passing the point P on Φ. Corresponding to any positive quantity ε, there exists a positive integer n, so large that as if two blocks R_1, R_2, representing G_1, G_2 in Lemma 2 respectively, have n-block in common, G_2 has at least one element within ε of the element of G_1 which lies at the midpoint of G_1.

Proof. No geodesic on Φ has on it two mutually conjugate points. A particular consequence of this result, as given in the theory of the calculus of variations, is that, if we vary the end points of a given geodesic segment, there exist further the geodesic segments joining these end points and varying continuously with the end points both in position and in length. As $\sum$ satisfies the non-conjugacy hypothesis, there is no more than one geodesic segment joining any two points on Φ, hence the length of the geodesic segment on Φ is a single valued continuous function of the position of those end points. In particular we have the following result:

There exists a finite upper limit to the set of the lengths of the geodesic segments which are mapped by f on the geodesic segments joining pairs of points lying on a closed set of points on Φ. Since each point of G_1 lies, by virtue of Lemma 2, within a geodesic distance d of some point of G_2, corresponding to any positive quantity ε, there exists a positive constant h so large that G_2 has at least one element within ε of the element of G_1 which lies at the midpoint of G_1. From the above result we see that there exists a positive integer n so large and corresponding to h.

Theorem 4. If $\sum$ is a closed orientable two-dimensional Riemannian manifold of genus $p>1$ such that no geodesic has on it a pair of conjugate points, then there exists a transitive geodesic ray represented by a symbolic transitive ray.

Proof. Let e be any element on $\sum$ and e^* be its image in S_0 by f and G^* be the geodesic segment whose mid element is e^*. We may assume that G^* is regular. We denote the n-block representing G^* by B. The geodesic segment including G^* is represented by $(n+2)$-block h_1Bh_2. Let h_1', h_2' be the proceeding symbols of a circular order to h_1, h_2 respectively and $h_1'Bh_2'$ be regular. If $h_1'Bh_2'$ is not regular, we replace h_1', h_2' by the proceeding symbols of a circular order to h_1', h_2' respectively. If $h_1'Bh_2'$ represents a segment G', G^* necessarily intersects G'. By virtue of Theorem 3, two geodesic rays with the same initial point can not be of the same type. As n tends to infinite, the geodesic ray represented by X and G^* necessarily intersects and can not be of the same type. By virtue of Lemma 3, corresponding to any positive quantity ε, the geodesic ray represented by X has at least one element within ε of e^*. This proves that there exists a transitive geodesic ray on $\sum$. By virtue of Lemma 1, there exists a transitive regular symbolic geodesic.

References

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