137. Probabilities on Inheritance in Consanguineous Families. XI

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VIII. Combinations through the most extreme consanguineous marriages

5. Distributions after successive consanguineous marriages

We have derived in §3 the probability of parent-descendant combination. Elimination of a type of parent leads to a corresponding *distribution of genotypes in a generation of descendant.* Namely, we have, for any $n\geqq 1$, a relation

$$\overline{A}_{(11;0)_{t-1}|n}(\xi\eta)=\sum\overline{A}_{ab}\mathfrak{k}_{t-1|n}(ab;\xi\eta).$$

In case $n=1$, we get, by actual computation,

$$\overline{A}_{(11;0)_{t-1}|1}(ii)=i-i(1-i)\boldsymbol{R}\frac{5+3\sqrt{5}}{5}\omega^t,$$

$$\overline{A}_{(11;0)_{t-1}|1}(ij)=\qquad 2ij\boldsymbol{R}\frac{5+3\sqrt{5}}{5}\omega^t.$$

The result shows that the distribution *deviates* from ordinary one. More precisely, there hold the relations

$$\overline{A}_{(11;0)_{t-1}|1}(\xi\eta)-\overline{A}_{\xi\eta}=2R(\xi\eta)\Big(1-\boldsymbol{R}\frac{5+3\sqrt{5}}{5}\omega^t\Big)$$

with $R(ii)=\frac{1}{2}i(1-i)$ and $R(ij)=-ij$. Namely, *any homozygous type increases while any heterozygous one decreases.* For instance, the values of the factor $1-\boldsymbol{R}(1+3\sqrt{5}/5)\omega^t$ are equal to 11/20, 23/40, 27/40, 117/160 etc. for $t=2, 3, 4, 5$ etc., respectively.

Though there exists a deviation in the first generation, it *vanishes out* soon in the next generation. In fact, as shown in §3, we have an identity $\mathfrak{k}_{t-1|n}=\kappa_n$ for any $n>1$, whence readily follows

$$\overline{A}_{(11;0)_{t-1}|n}(\xi\eta)=\overline{A}_{\xi\eta}.$$

By the way, it would be noted that the frequency of gene A_i in the first generation is given by

$$\overline{A}_{(11;0)_{t-1}|1}(ii)+\frac{1}{2}\sum_{b\neq i}\overline{A}_{(11;0)_{t-1}|1}(ib)=i.$$

Consequently, the random matings *within* the generation produce also the distribution coincident with the original one, i. e. $\overline{A}_{ii}=i^2$, $\overline{A}_{ij}=2ij$.

6. Descendants combinations

By eliminating a type of parent from a parent-descendants combination, we get a corresponding *descendants combination*. Its probability is namely defined, for $\mu, \nu \geqq 1$, by an equation

$$\mathfrak{s}_{t-1|\mu\nu}(\xi_1\eta_1, \xi_2\eta_2) \equiv \sigma_{(11;0)_{t-1}|\mu\nu}(\xi_1\eta_1, \xi_2\eta_2) = \sum \overline{A}_{ab}\mathfrak{t}_{t-1|\mu\nu}(ab; \xi_1\eta_1, \xi_2\eta_2).$$

In *case* $\mu=\nu=1$, we get, by actual computation, the following results:

$$\mathfrak{s}_{t-1|11}(ii, ii) = i + i(1-i)\left\{\boldsymbol{R}\frac{9+4\sqrt{5}}{5}\omega^t + \frac{1}{2^t} - \frac{1}{5}\frac{1}{4^t} + i\left(-\frac{2}{2^t} + \frac{1}{4^t}\right) - i^2\frac{1}{4^t}\right\},$$

$$\mathfrak{s}_{t-1|11}(ii, ik) = ik\left\{\boldsymbol{R}\frac{3+\sqrt{5}}{5}\omega^t - \frac{1}{2^t} + \frac{2}{5}\frac{1}{4^t} + i\left(\frac{2}{2^t} - \frac{2}{4^t}\right) + i^2\frac{2}{4^t}\right\},$$

$$\mathfrak{s}_{t-1|11}(ii, kk) = ik\left\{\boldsymbol{R}\frac{1}{5}\omega^t - \frac{1}{5}\frac{1}{2^t} + \frac{2}{15}\frac{1}{4^t} - \frac{2}{15}\frac{1}{(-8)^t}\right.$$
$$\left. + (i+k)\left(\frac{1}{5}\frac{1}{2^t} - \frac{1}{3}\frac{1}{4^t} + \frac{2}{15}\frac{1}{(-8)^t}\right) + ik\frac{1}{4^t}\right\},$$

$$\mathfrak{s}_{t-1|11}(ii, hk) = ihk\left\{\frac{2}{5}\frac{1}{2^t} - \frac{2}{3}\frac{1}{4^t} + \frac{4}{15}\frac{1}{(-8)^t} + i\frac{2}{4^t}\right\},$$

$$\mathfrak{s}_{t-1|11}(ij, ij) = ij\left\{\boldsymbol{R}\frac{4+4\sqrt{5}}{5}\omega^t - \frac{8}{5}\frac{1}{2^t} + \frac{8}{15}\frac{1}{4^t} + \frac{4}{15}\frac{1}{(-8)^t}\right.$$
$$\left. + (i+j)\left(\frac{8}{5}\frac{1}{2^t} - \frac{4}{3}\frac{1}{4^t} - \frac{4}{15}\frac{1}{(-8)^t}\right) + ij\frac{4}{4^t}\right\},$$

$$\mathfrak{s}_{t-1|11}(ij, ik) = ijk\left\{\frac{8}{5}\frac{1}{2^t} - \frac{4}{3}\frac{1}{4^t} - \frac{4}{15}\frac{1}{(-8)^t} + i\frac{4}{4^t}\right\},$$

$$\mathfrak{s}_{t-1|11}(ij, hk) = 4ijhk\frac{1}{4^t}.$$

In *case* $\mu=1<\nu$, we get the following formula:

$$\mathfrak{s}_{t-1|1\nu}(\xi_1\eta_1, \xi_2\eta_2)$$
$$= \overline{A}_{(11;0)_{t-1}|1}(\xi_1\eta_1)\overline{A}_{\xi_2\eta_2} + 2^{-\nu+2}\{\mathfrak{s}_{t-1|12}(\xi_1\eta_1, \xi_2\eta_2) - \overline{A}_{(11;0)_{t-1}|1}(\xi_1\eta_1)\overline{A}_{\xi_2\eta_2}\}.$$

Hence, it is sufficient to determine the values of $\mathfrak{s}_{t-1|12}$. They will be set out in the following lines:

$$\mathfrak{s}_{t-1|12}(ii, ii) = i^2 + i^2(1-i)\left\{-\boldsymbol{R}\frac{15+7\sqrt{5}}{10}\omega^t + \frac{1}{2}\frac{1}{2^t} - i\frac{1}{2^t}\right\},$$

$$\mathfrak{s}_{t-1|12}(ii, ik) = ik + ik\left\{-\boldsymbol{R}\frac{15+7\sqrt{5}}{10}\omega^t + \frac{1}{2}\frac{1}{2^t} + i\left(\boldsymbol{R}\frac{10+4\sqrt{5}}{5}\omega^t - \frac{2}{2^t}\right) + i^2\frac{2}{2^t}\right\},$$

$$\mathfrak{s}_{t-1|12}(ii, kk) = ik^2\left\{\boldsymbol{R}\frac{5+\sqrt{5}}{10}\omega^t - \frac{1}{2}\frac{1}{2^t} + i\frac{1}{2^t}\right\},$$

$$\mathfrak{s}_{t-1|12}(ii, hk) = ihk\left\{\boldsymbol{R}\frac{5+\sqrt{5}}{5}\omega^t - \frac{1}{2^t} + i\frac{2}{2^t}\right\},$$

$$\mathfrak{s}_{t-1|12}(ij, ii) = i^2j\left\{\boldsymbol{R}\frac{5+3\sqrt{5}}{5}\omega^t - \frac{1}{2^t} + i\frac{2}{2^t}\right\},$$

$$\mathfrak{s}_{t-1|12}(ij, ij) = ij\left\{(i+j)\left(\boldsymbol{R}\frac{5+3\sqrt{5}}{5}\omega^t - \frac{1}{2^t}\right) + ij\frac{4}{2^t}\right\},$$

$$\mathfrak{s}_{t-1|12}(ij, ik) = ijk\left\{\boldsymbol{R}\frac{5+3\sqrt{5}}{5}\omega^t - \frac{1}{2^t} + i\frac{4}{2^t}\right\},$$

$$\mathfrak{s}_{t-1|12}(ij, kk) = ijk^2\frac{2}{2^t},$$

$$\mathfrak{s}_{t-1|12}(ij, hk) = ijhk\frac{4}{2^t}.$$

In *case* $\mu, \nu>1$ with $\lambda=\mu+\nu-1$, we get the following formula:

$$\mathfrak{s}_{t-1|\mu\nu}(\xi_1\eta_1, \xi_2\eta_2)=\overline{A}_{\xi_1\eta_1}\overline{A}_{\xi_2\eta_2}+2^{-\lambda+3}\{\mathfrak{s}_{t-1|22}(\xi_1\eta_1, \xi_2\eta_2)-\overline{A}_{\xi_1\eta_1}\overline{A}_{\xi_2\eta_2}\}.$$

Hence, it suffices to determine the values of $\mathfrak{s}_{t-1|22}$, which are set out as follows:

$$\mathfrak{s}_{t-1|22}(ii, ii)=i^3-i^3(1-i)\boldsymbol{R}\frac{5+2\sqrt{5}}{5}\omega^t,$$

$$\mathfrak{s}_{t-1|22}(ii, ik)=i^2k-i^2k(1-2i)\boldsymbol{R}\frac{5+2\sqrt{5}}{5}\omega^t,$$

$$\mathfrak{s}_{t-1|22}(ii, kk)=i^2k^2\boldsymbol{R}\frac{5+2\sqrt{5}}{5}\omega^t,$$

$$\mathfrak{s}_{t-1|22}(ii, hk)=2i^2hk\boldsymbol{R}\frac{5+2\sqrt{5}}{5}\omega^t,$$

$$\mathfrak{s}_{t-1|22}(ij, ij)=ij(i+j)-ij(i+j-4ij)\boldsymbol{R}\frac{5+2\sqrt{5}}{5}\omega^t,$$

$$\mathfrak{s}_{t-1|22}(ij, ik)=ijk-ijk(1-4i)\boldsymbol{R}\frac{5+2\sqrt{5}}{5}\omega^t,$$

$$\mathfrak{s}_{t-1|22}(ij, hk)=4ijhk\boldsymbol{R}\frac{5+2\sqrt{5}}{5}\omega^t.$$

7. Limit behaviors of the probabilities

Most of the expressions derived in the present chapter involve a consanguineous generation-number t as a parameter and are of a form linear in 1, 2^{-t}, 4^{-t}, $(-4)^{-t}$, $(-8)^{-t}$, ω^t and $\tilde{\omega}^t$. Since

$$\omega=\frac{1+\sqrt{5}}{4}=0.809\cdots \quad \text{and} \quad \tilde{\omega}=\frac{1-\sqrt{5}}{4}=-0.309\cdots,$$

any term factored by ω^t majorates ultimately those factored by others except 1, while a term factored by ω^t itself tends to zero as $t\to\infty$.

A brief account will be given with respect to asymptotic behaviors of expressions involving ω^t. We introduce sequences of rational numbers, $\{c_t\}$ and $\{d_t\}$, defined by

$$\omega^t=c_t+d_t\sqrt{5} \qquad (t=0, 1, 2, \ldots).$$

As readily seen, they are given by

$$c_t=\frac{1}{2}(\omega^t+\tilde{\omega}^t), \qquad d_s=\frac{1}{2\sqrt{5}}(\omega^t-\tilde{\omega}^t)$$

and satisfy the recurrence relations

$$c_t=\frac{1}{4}(c_{t-1}+5d_{t-1}), \qquad d_t=\frac{1}{4}(c_{t-1}+d_{t-1}).$$

If we separate the quantities c's and d's from the last system, we obtain difference equations of the second order satisfied by them, i. e.

$$c_t=\frac{1}{2}c_{t-1}+\frac{1}{4}c_{t-2}, \qquad d_t=\frac{1}{2}d_{t-1}+\frac{1}{4}d_{t-2}.$$

Though the last two equations are of the same form, the initial conditions are different. We may put here

$$c_{-1}=-1, \qquad c_0=1; \qquad d_{-1}=1, \qquad d_0=0;$$

the last equations are then valid for $t \geqq 1$.

For sufficiently large t, we have asymptotically

$$c_t \sim \frac{1}{2}\omega^t, \qquad d_t \sim \frac{1}{2\sqrt{5}}\omega^t.$$

Consequently, *for any rational numbers p and q, we get, for sufficiently large t, an asymptotic relation*

$$R(p+q\sqrt{5})\omega^t = pc_t + 5qd_t \sim \frac{1}{2}(p+q\sqrt{5})\omega^t.$$

Now, the asymptotic behaviors, as $t \to \infty$, of several probabilities derived in the present chapter can be readily concluded from their own expressions. For instance, we get for a limit probability defined by

$$\mathfrak{e}_\infty(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2) \equiv \lim_{t\to\infty} \mathfrak{e}_t(\alpha\beta, \gamma\delta; \xi_1\eta_1, \xi_2\eta_2)$$

the following values:

$$\mathfrak{e}_\infty(ii, ii; ii, ii) = 1, \qquad \mathfrak{e}_\infty(ii, ik; ii, ii) = \tfrac{3}{4},$$
$$\mathfrak{e}_\infty(ii, ik; kk, kk) = \tfrac{1}{4}, \qquad \mathfrak{e}_\infty(ii, kk; ii, ii) = \tfrac{1}{2},$$
$$\mathfrak{e}_\infty(ik, ik; ii, ii) = \tfrac{1}{2}.$$

The values for combinations other than those essentially exhausted here are equal to zero.

The remaining limit probabilities such as

$$\mathfrak{e}_{\infty|n}(\alpha\beta, \gamma\delta; \xi\eta), \qquad \mathfrak{f}_{\infty|n}(\alpha\beta; \xi\eta), \qquad \mathfrak{f}_{\infty|\mu\nu}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2)$$
$$\overline{A}_{\langle 11;0\rangle\infty|n}(\xi\eta), \qquad \mathfrak{s}_{\infty|\mu\nu}(\xi_1\eta_1, \xi_2\eta_2) \quad \text{etc.}$$

can also be treated simply. For instance, we get

$$\overline{A}_{\langle 11;0\rangle\infty|1}(ii) = i, \qquad \overline{A}_{\langle 11;0\rangle\infty|1}(ij) = 0.$$

Thus, *repetition of the most extreme consanguineous marriages causes the disappearance of any heterozygous type.* Moreover, *the frequency of any homozygous type in the limit distribution coincides just with that of its constituting gene.*

Corrections to Yûsaku Komatu and Han Nishimiya: "Probabilities on Inheritance in Consanguineous Families. VII"

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Corrections should be made to the values of

$$\mathfrak{Y}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2) = \sum S(\alpha\beta; ab)\kappa(ab; \xi_1\eta_1, \xi_2\eta_2)$$

listed in the above paper (pp. 214–242).

In fact, the values given in the paper were those for the quantity $\frac{1}{4}S(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2)-\mathfrak{Y}(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2)$. The corrected values for $\mathfrak{Y}(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2)$ are listed in the following lines:*)

$$\mathfrak{Y}(ii;\, ii,\, ii) = \tfrac{1}{16}i(1-i)(1+i)(1-2i), \qquad \mathfrak{Y}(ii;\, ii,\, ig) = -\tfrac{1}{8}i^2g(1-2i),$$
$$\mathfrak{Y}(ii;\, ii,\, gg) = -\tfrac{1}{16}ig^2(1-2i), \qquad \mathfrak{Y}(ii;\, ii,\, fg) = -\tfrac{1}{8}ifg(1-2i),$$
$$\mathfrak{Y}(ii;\, ik,\, ik) = \tfrac{1}{16}k(1-4i+k+i^2-3ik+8i^2k),$$
$$\mathfrak{Y}(ii;\, ik,\, kk) = -\tfrac{1}{16}k^2(1-i+k-4ik),$$
$$\mathfrak{Y}(ii;\, ik,\, ig) = \tfrac{1}{16}kg(1-3i+8i^2), \qquad \mathfrak{Y}(ii;\, ik,\, kg) = -\tfrac{1}{16}kg(1-i+2k-8ik),$$
$$\mathfrak{Y}(ii;\, ik,\, gg) = -\tfrac{1}{16}kg^2(1-4i), \qquad \mathfrak{Y}(ii;\, ik,\, fg) = -\tfrac{1}{8}kfg(1-4i),$$
$$\mathfrak{Y}(ii;\, kk,\, kk) = -\tfrac{1}{16}k^2(1+k)(1-2k), \qquad \mathfrak{Y}(ii;\, kk,\, kg) = \tfrac{1}{16}k^2g(1+4k),$$
$$\mathfrak{Y}(ii;\, kk,\, gg) = \tfrac{1}{8}k^2g^2, \qquad \mathfrak{Y}(ii;\, kk,\, fg) = \tfrac{1}{4}k^2fg,$$
$$\mathfrak{Y}(ii;\, hk,\, hk) = -\tfrac{1}{16}hk(2-h-k-8hk), \qquad \mathfrak{Y}(ii;\, hk,\, kg) = \tfrac{1}{16}hkg(1+8k),$$
$$\mathfrak{Y}(ii;\, hk,\, fg) = \tfrac{1}{2}hkfg;$$
$$\mathfrak{Y}(ij;\, ii,\, ii) = \tfrac{1}{32}i(1+i)(1-2i)^2, \qquad \mathfrak{Y}(ij;\, ii,\, ij) = -\tfrac{1}{32}i^2(1+i+j-8ij),$$
$$\mathfrak{Y}(ij;\, ii,\, jj) = -\tfrac{1}{32}ij(i+j-4ij), \qquad \mathfrak{Y}(ij;\, ii,\, ig) = -\tfrac{1}{32}i^2g(1-8i),$$
$$\mathfrak{Y}(ij;\, ii,\, jg) = -\tfrac{1}{32}ig(i+2j-8ij), \qquad \mathfrak{Y}(ij;\, ii,\, gg) = -\tfrac{1}{32}ig^2(1-4i),$$
$$\mathfrak{Y}(ij;\, ii,\, fg) = -\tfrac{1}{16}ifg(1-4i),$$
$$\mathfrak{Y}(ij;\, ij,\, ij) = \tfrac{1}{32}(i+j+i^2+j^2-8ij-2i^2j-2ij^2+16i^2j^2),$$
$$\mathfrak{Y}(ij;\, ij,\, ig) = -\tfrac{1}{32}g(i-j+2i^2-2ij-16i^2j),$$
$$\mathfrak{Y}(ij;\, ij,\, gg) = -\tfrac{1}{32}g^2(i+j-8ij), \qquad \mathfrak{Y}(ij;\, ij,\, fg) = -\tfrac{1}{16}fg(i+j-8ij),$$
$$\mathfrak{Y}(ij;\, ik,\, ik) = \tfrac{1}{31}k(1-6i+k+2i^2-2ik+16i^2k),$$
$$\mathfrak{Y}(ij;\, ik,\, jk) = -\tfrac{1}{32}(i+j-2ij+2ik+2jk-16ijk),$$
$$\mathfrak{Y}(ij;\, ik,\, kk) = -\tfrac{1}{32}k^2(1-2i+k-8ik), \qquad \mathfrak{Y}(ij;\, ik,\, ig) = \tfrac{1}{32}kg(1-2i+16i^2),$$
$$\mathfrak{Y}(ij;\, ik,\, jg) = -\tfrac{1}{16}kg(i+j-8ij), \qquad \mathfrak{Y}(ij;\, ik,\, kg) = -\tfrac{1}{32}(1-2i+2k-16ik),$$
$$\mathfrak{Y}(ij;\, ik,\, gg) = -\tfrac{1}{32}kg^2(1-8i), \qquad \mathfrak{Y}(ij;\, ik,\, fg) = -\tfrac{1}{16}kfg(1-8i).$$