123. *Relations between Harmonic Dimensions*

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M. Ozawa proposed the following problem:[1] Let F be a null-boundary Riemann surface with one ideal component and D be a non compact domain which has a finite number of analytic curves as its relative boundary. Denote by $\overset{**}{\dim} D$ the number of linearly independent generalized Green's functions. (See the definition given below.) Let F_0 be a compact disc which has no common point with D. Then we have the relation:

$$\overset{**}{\dim} D \leqq \dim (F-F_0)\,?$$

It is the purpose of this article to give a solution to the problem.

Let F be an abstract Riemann surface, $\{F_n\}$ an exhaustion of F and D a non compact domain of F, whose relative boundary ∂D[2] consists of at most enumerable number of analytic curves clustering nowhere in F. Let $\{p_i\}$ be a sequence of points in D, such that $\{p_i\}$ converges to the boundary of F, and let $G(z, p_i)$ be the Green's function of D with pole at p_i. Take a subsequence of $\{G(z, p_i)\}$ which converges uniformly to a non-constant function $G(z, \{p'_i\})$ which we call *generalized Green's function.* Denote by F_0 a compact disc which has no common point with D and let $G_{F-F_0}(z, p_0)$ be the Green's function of $F-F_0$, where p_0 is an inner point of D. In this case, it is clear that $\infty > \overline{\lim_i}\, G_{F-F_0}(p_i, p_0) \geqq \lim_i G(p_i, p_0) > 0$. $\infty > \lim_i G_{F-F_0}(p'_i, z) = G_{F-F_0}(z, \{p'_i\}) \geqq G(z, \{p'_i\})$ for every point z, whence $G(z, \{p'_i\})$ is finite in $D \cap F_n$ $(n=1, 2, 3, \ldots)$. Put $D^N(\{p'_i\}) = \mathcal{E}\{z; G(z, \{p'_i\}) \geqq N\}$. We denote by $\hat{D}^N(p_0)$ the symmetric surface of $D^N(p_0)(=\mathcal{E}\{z; G(z, p_0) \geqq N\})$ with respect to $\partial D^N(p_0)$. Then $D^N(p_0) + \hat{D}^N(p_0)$ is a null-boundary[3] Riemann surface.

Lemma.

$$\int_{\partial D^N(\{p'_i\})} \frac{\partial G(z, \{p'_i\})}{\partial n} ds = \delta(\{p'_i\}) \leqq 2\pi.$$

Proof. Denote by $\{D_n\}$ the exhaustion of D. Since $G(z, p_i)$ is bounded outside a neighbourhood v of p_i, we have $D_{D^N(p_i)-v}(G(z, p)) < \infty$

1) At the annual meeting of the Mathematical Society of Japan held on the 30th of May, 1954. M. Ozawa: On harmonic dimensions I and II, to appear in Kōdai Mathematical Seminar Reports.

2) In this article we denote by ∂G the relative boundary of G with respect to F.

3) Z. Kuramochi: Harmonic measures and capacity of a subset of the ideal boundary of abstract Riemann surface, to appear in the Proceedings of the Japan Academy.

by Nevanlinna's theorem.[4] It follows

$$\lim_{n=\infty}\int_{D^N(p_i)\cap\partial D_n}\frac{\partial(G(z,p_i)}{\partial n}ds=0,$$

and since

$$\frac{\partial G(z,p_i)}{\partial n}\geqq 0 \quad \text{on} \quad \partial D^N(p_i) \quad \text{and} \int_{D_n\cap\partial D^N(p_i)}\frac{\partial G(z,p)}{\partial n}ds<\infty,$$

we have

$$2\pi=\int_{D\partial\cap D^M(p_i)}\frac{\partial G(z,p)}{\partial n}ds=\int_{\partial D^N(p_i)}\frac{\partial G(z,p_i)}{\partial n}ds,$$

by taking M so large that $D^M(p_i)$ is contained in a compact subset of D. For given D_n, since $\{G(z,p_i')\}$ converges to $G(z,\{p_i'\})$, there exists a number $m(n)$ such that

$$\int_{D_n\cap\partial D^N(\{p'_i\})}\frac{\partial G}{\partial n}(z,\{p_i'\})ds\leqq\int_{D_n\cap\partial D^N(\{p'_i\})}\frac{\partial G(z,p_m')}{\partial n}ds+\varepsilon\leqq\int_{D_n\cap\partial D^N(p'_m)}\frac{\partial G(z,p_m')}{\partial n}ds$$

$$+2\varepsilon\leqq\int_{D\cap\partial D^N(p'_m)}\frac{\partial G(z,p_m')}{\partial n}ds+3\varepsilon\leqq 2\pi+3\varepsilon(m'\geqq m(n)).$$

Let $m\to\infty$ and then $n\to\infty$. We have

$$\int_{D\cap\partial D^N(p'_m)}\frac{\partial G(z,\{p_i'\})}{\partial n}ds=\delta(\{p_i'\})\leqq 2\pi \qquad \text{for every } N \qquad (N>0).$$

Let $S(z)$ be a positive harmonic function in D such that $S(z)=0$ on ∂D, $S(z)$ is finite in $D\cap F$ and $\int_{\partial(D\cap F_n)}\frac{\partial S(z)}{\partial n}ds<\delta(S)<\infty$ $(n=1, 2, 3, \dots)$. By the same method used in lemma, we can prove that our $G(z,\{p_i'\})$ satisfies the above conditions.

Extremisation. Define harmonic functions $V_n^N(z)$ $(n=1,2,\dots)$ such that $V_n^N(z)$ is harmonic in $(F_n-F_0-D^N(S))$, $V_n^N(z)=0$ on $\partial F_0+\{\partial F_n\cap(F-D)\}$, $V_n^N(z)=S(z)=N$ on $\partial D^N(S)$ and $V_n^N(z)=S(z)$ on $\partial F_n\cap(D-D^N(S))$. Then $V_{n+i}^N(z)\geqq V_n^N(z)$ and $V_n^{N'}(z)\geqq V_n^N(z)$ $(N'\geqq N)$. Put $V^N(z)=\lim_n V_n^N(z)$ and $\lim_N V^N(z)=V^*(z)$. We see easily that

$$\frac{\partial V_n^N(z)}{\partial n}\leqq\frac{\partial S(z)}{\partial n} \qquad \text{on} \qquad \partial D^N(S)+\{(D-D^N(S))\cap\partial F_n\}$$

and

$$\frac{\partial V_n^N(z)}{\partial n}\leqq 0 \qquad \text{on} \qquad \partial F_n\cap(F-D).$$

Therefore

$$\infty>\delta(S)=\int_{\partial(D\cap F_n)}\frac{\partial S(z)}{\partial n}ds\geqq\int_{(\partial D^N\cap F_n)+\{\partial F_n\cap F-D^N(S)\}}\frac{\partial V_n^N(z)}{\partial n}ds$$

$$=\int_{\partial F_0}\frac{\partial V_n^N(z)}{\partial n}ds \qquad \text{for every } N \text{ and } n.$$

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4) R. Nevanlinna: Quadratisch integrierbare Differentiale auf einer Riemannschen Manigfaltigkeit, Ann. Acad. Sci. Fenn., A, I, 1 (1941).

Hence $0<\lim_N V^N(z)=V^*(z)<\infty$.

Let $W^n(z)$ be harmonic in F_n-F_0 such that $W^n(z)=0$ on $\partial F_0+\partial F_n\cap(F-D)$ and $W^n(z)=S(z)$ on $\partial F_n\cap D$. Take a subsequence of $\{W^n(z)\}$ which converges uniformly in wider sense in D and denote by $S_{ex}(z)$ its limit function. $S_{ex}(z)$ may depend on the exhaustion. Since $S(z)\leqq V^*(z)$ on $\partial F_n\cap D$, it is clear that

$$S_{ex}(z)\leqq V^*(z). \tag{1}$$

We say that this $S_{ex}(z)$ is obtained from $S(z)$ by "extremisation".

Theorem 1. The extremisation does not depend on the exhaustion.

Proof. We see, for any positive and harmonic function $S(z)$ which vanishes on ∂D, that the following inequality holds

$$S_{ex}(z)\geqq S(z). \tag{2}$$

We see that

$$S^1(z)\geqq S^2(z)\geqq 0 \quad \text{implies} \quad S^1_{ex}(z)\geqq S^2_{ex}(z). \tag{3}$$

For any function $S(z)$ which is positive and harmonic in $F-F_0$,

$$S_{ex}(z)\leqq S(z). \tag{4}$$

For another exhaustion $\{F'_n\}$, define $S_{ex'}(z)$. We have by (2)

$$S(z)\leqq S_{ex}(z), \quad \text{which implies} \quad S_{ex'}(z)\leqq S_{ex\,ex'}(z),$$

and by (4) and (3) we have

$$S_{ex\,ex'}(z)\leqq S_{ex}(z).$$

Therefore

$$S_{ex}(z)=S_{ex'}(z).$$

Inverse Extremisation. Let $U(z)$ be a positive harmonic function in $F-F_0$ such that $U(z)=0$ on ∂F_0. Let $U^n(z)$ be harmonic function in $D\cap F_n$ such that $U^n(z)=0$ on ∂D and $U^n(z)=U(z)$ on $\partial F_n\cap(F-F_0)\cap D$. Since $\{U^n(z)\}$ is a normal family, there exists a subsequence $\{U^{n'}(z)\}$ which converges in wider sense in D to $U_{in\,ex}(z)$. As above, we may prove that the limit function, which we denote by $U_{in\,ex}(z)$, does not depend upon the exhaustion. We say that $U_{in\,ex}(z)$ is obtained from $U(z)$ by "inverse extremisation".

Theorem 2.

$$S(z)=(S_{ex}(z))_{in\,ex}.$$

Proof. Let $U(z)=S_{ex}(z)$ and put $S_{ex}(z)=\lim_n W^n(z)$. Then

$$S_{ex}(z)\geqq W^n(z).$$

Now

$$\begin{aligned} S_{ex}(z)-U^n(z)&=S_{ex}(z) && \text{on} \quad \partial D,\\ W^n(z)-S(z)&=W^n(z) && \text{on} \quad \partial D,\\ S_{ex}(z)-U^n(z)&=0 && \text{on} \quad \partial F_n\cap D,\\ W^n(z)-S(z)&=0 && \text{on} \quad \partial F_n\cap D. \end{aligned}$$

Therefore $S_{ex}(z)-U^n(z)\geqq W^n(z)-S(z)$, and letting $n\to\infty$. We have $S_{ex}(z)-U_{in\,ex}(z)\geqq S_{ex}(z)-S(z)$ and $U_{in\,ex}(z)\leqq S(z)$.

On the other hand, it is clear that $(S_{ex}(z))_{in\ ex} \geqq S(z) \geqq 0$. Thus we have

$$S(z) = (S_{ex}(z))_{in\ ex}.$$

Property of $S_{ex}(z)$. Let $\hat{V}^n(z)$ $(\check{V}^n(z))$ be harmonic function in $F-F_0$ such that $\hat{V}^n(z)=0$ on $(\partial F_n \cap D)+\partial F_0$, $\hat{V}^n(z)=S_{ex}(z)$ on $\partial F_n \cap (F-D)$, $\check{V}^n(z)=0$ on $\partial F_n \cap (F-D)+\partial F_0$ and $\check{V}^n(z)=S_{ex}(z)$ on $\partial F_n \cap D$. Then $\hat{V}^n(z)+\check{V}^n(z)=S_{ex}(z)$ and $S_{ex}(z) \geqq \lim_n \check{V}^n(z) \geqq S_{ex}(z)$. Therefore

$$\lim_n \hat{V}^n(z)=0.$$

We have easily next

Corollary 1. If $S^i(z)$ *and* $S^j(z)$ *are linearly independent in* D, *then* $S^i_{ex}(z)$ *and* $S^j_{ex}(z)$ *are linearly independent in* $F-F_0$.

We have, from the property of $S_{ex}(z)$, next

Corollary 2. If D_1 *and* D_2 *are two non compact domains such that* $D_1 \cap D_2 = 0$ *and* $S^i_{D_1}(z)$ *and* $S^j_{D_2}(z)$ *are functions as above on* D_1 *and* D_2 *respectively, then* $S^i_{D_1\,ex}(z)$ *and* $S^j_{D_2\,ex}(z)$ *are linearly independent.*

From the above corollaries, we have next

Corollary 3. Denote by $\overset{*}{\dim} D$ *the number of* $S(z)$ *satisfying the above three conditions which are linearly independent and by* $\dim(F-F_0)$ *the number of harmonic functions which are linearly independent and vanish on* ∂F_0. *Then we have*

$$\overset{*}{\dim} D \leqq \dim(F-F_0),$$ [5]

$$\overset{*}{\dim} D_1 + \overset{*}{\dim} D_2 \leqq \dim(F-F_0).$$

We shall apply the result to the planer surface.

Let U: $|z|<1$ be a unit circle and E be a closed set which has $z=0$ as its limit point. We denote by $G^i(z)$ $(i=1, 2, \ldots)$ the limit function of a uniformly convergent sequence $G(z, p^i_j)$ $(j=1, 2, \ldots)$ of Green's function of $U-E$, where $\lim_j p^i_j = (z=0)$. Then all $G^i(z)$ are linearly dependent, because $U-(z=0)$ has only one minimal function $-\log|z|$ vanishing on $|z|=1$. We see that $\int_{\partial(U-E)} \frac{\partial G^i(z)}{\partial n} ds < 2\pi$ except at most one, and the other function loose their mass when $\{p^i_j\}$ tend to $z=0$. Such assertion holds either when E is or is not so thick distributed that $G^i(z) \equiv 0$ $(i=1, 2, \ldots)$.

Theorem 3. Let $D^N_i = \mathcal{E}\{z; G^i(z) \geqq N\}$, *where* $G^i(z)$ *is a minimal positive function like a Green's function. Then for any fixed* N, $(D-\sum_i D^N_i)$ *is compact or* $\lim_j G(z, q_j)=0$ *for every sequence* $\{q_j\} \in (D-\sum_i D^N_i)$ *which converges to the boundary.*

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5) It is clear that $\overset{**}{\dim} D \leqq \overset{*}{\dim} D$. Thus the corollary gives the answer to the problem mentioned at the top of this note.

Proof. $\int_{\partial D^N(q_j)} \frac{\partial G(z, q_j)}{\partial n} ds = 2\pi$, then, by Fatou's lemma,

$$0 \leqq \delta = \int_{\partial D^N(\{q_j\})} \lim_j \frac{\partial G(z, q_j)}{\partial n} ds \leqq \lim_j \int \frac{\partial G(z, q_j)}{\partial n} ds = 2\pi.$$

We see easily that δ is zero if and only if $G(z, \{q_j\})=0$. And we can prove the validity of Green's formula for $G(z, \{q_j\})$ by the same method used in Lemma. Thus the theorem may be proved similarly as in the previous paper.[6)]

Remark. Theorem 1[7)] to corollary 3 are valid for function $K(z, \{p_i\})$ defined by $\frac{G(z, \{p_i\})}{G(p_0, \{p_i\})}$, when $K_{ex}(z, \{p_i\}) < \infty$.

For example, we have the result.

Let $\{p_i\}$ be a sequence in D which converges to the boundary of F, and if there exists a constant such that

$$\delta \geqq \overline{\lim_i} \frac{G_{F-F_0}(p_i, p_0)}{G(p_i, p_0)},$$

where $G_{F-F_0}(z, p_0)$ and $G(z, p_0)$ are Green's functions of $F-F_0$ and D respectively.

Proof.

$$\frac{K(z, p_i)}{K_{F-F_0}(z, p_i)} = \frac{G(z, p_i)}{G(p_0, p_i)} \frac{G_{F-F_0}(p_0, p_i)}{G_{F-F_0}(z, p_i)} \leqq \delta,$$

$K_{F-F_0}(z, p) \geqq \frac{1}{\delta} K(z, p_i)$ for sufficiently large i.

Hence

$$\infty > K_{F-F_0}(z, \{p_i\}) \geqq \frac{1}{\delta} K_{ex}(z, \{p_i\}).$$

6) Z. Kuramochi: An example of a null-boundary Riemann surface, Osaka Math. *Journ.*, **6** *(1954).*

7) Through this article we do not assume that F is a null-boundrry Riemann surface.