119. On Newman Algebra. II

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Introduction

In our earlier paper "On Newman Algebra"[1] (we shall refer to the paper as I) we have considered the Newman algebra in a somewhat different sense from the usual one, and given a postulate-set for this algebraic system. Namely, in Newman algebra in the usual sense[2], the associative law for multiplication does not always hold, while this law holds in the system considered in I. We shall continue to use the term *Newman algebra* in the same sense as in I, and propose to give another postulate-set for the system. The postulate-set given in I will be denoted as Set I*, and the new set as Set II*. Set II* is obtained from Set I* by replacing the commutative law for addition

I*. 3. $a+b=b+a$

by a more restricted one

II*. 8′. $a+b'b=b'b+a$.

In §1 we shall give a complete list of postulates of Set II*, and prove the equivalence of Sets I* and II*. In I, we have obtained as a by-product the postulate-sets I and II characterizing respectively the Boolean lattice and the Boolean ring with unity. Correspondingly we shall obtain here new postulate-sets III and IV. In §2 we shall give a proof for the associative law for addition $(a+b)+c=a+(b+c)$, different from the one given in I, and in §3 independence proofs for our Set II* will be given. As the Sets I, II included the Set I* in I, so that the independence proofs for the Sets I, II implied that of Set I*, so the Sets III, IV include our Set II*, and the independence proofs for the Sets III, IV imply that of Set II*. Now the independence systems for the Sets I, II serve also as such for the Sets III, IV. But we shall show that we can construct the simpler independence systems for Postulate 5 of Sets III, IV than those given in I. These systems show also the independence of Postulate 5 of Sets I*, I, II.

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1. Postulates of Set II*

Our postulates are the propositions below on a *class* K, a *binary operation* $+$, a *binary operation* $\times$, and a *unary operation* $'$ (in the postulates that are not existence postulates supply the condition: *if the elements indicated are in* K). It is to be remarked that the unary operation $'$ is not required to be single-valued in our postulates.

Set II*

System $(K, +, \times, ')$

1. K is not empty.
2. If $a,\ b \in K$, then an element $a+b \in K$ is uniquely determined.
4. If $a,\ b \in K$, then an element $a \times b \in K$ is uniquely determined.

(For the sake of brevity we shall write ab for $a \times b$.)

5. $a(bc)=b(ca)$.
6. $a(b+c)=ab+ac$.
7. To each $a \in K$ corresponds at least one $a' \in K$.
8. $a+b'b=a$.

8′. $a+b'b=b'b+a$.

9. $a(b'+b)=a$.

Obviously the postulates of Set II* follow from those of the Set I*; we obtain the Postulate 8′ of Set II* from the Postulates 3 and 8 of Set I*. To prove the equivalence of Sets I* and II* it remains only to show that the Postulate 3 of Set I* can be deduced from Set II*. This follows from the following theorems and lemmas.

$(T0)$ $a'a=aa'$.

$(T1)$ $aa=a$.

$(T2)$ $ab=ba$.[3]

$(T3)$ $(a+b)c=ac+bc$.

$(T4)$ $a(bc)=(ab)c$.

As to the proofs of $(T0)$-$(T4)$, we refer to I. [Cf. 1): $T0$, $T1$, $T2$, $T8$, $T10$. We have only to use Postulate 8′ instead of Postulate 3.]

$(L1)$ $ab'+(ab)b'=(ab)b'+ab'$.

Proof. $ab'+(ab)b'=(ab')b'+(ab)b'=\{a(b'+b)\}b'=ab'$ $=a(bb'+b')=a(bb')+ab'=(ab)b'+ab'$ by $(T1)$-$(T4)$, $(T3)$-6, 9, 8-8′-$(T0)$, 6, $(T4)$.

$(L2)$ $ab+(ab)b=(ab)b+ab$.

Proof. $ab+(ab)b=a(bb)+a(bb)=(ab)b+ab$ by $(T1)$-$(T4)$, $(T4)$-$(T1)$.

$(L3)$ $a+ab=ab+a$.

Proof. $a+ab=(a+ab)(b'+b)=\{ab'+(ab)b'\}+\{ab+(ab)b\} = \{(ab)b' +ab'\}+\{(ab)b+ab\}=(ab+a)(b'+b)=ab+a$ by 9, 6-$(T3)$, $(L1)$-$(L2)$,

($T3$)-6, 9.

($T5$) $a+b=b+a$.

Proof. $a+b=(a+b)(b'+b)=(ab'+bb')+(ab+bb)$

$$=(b'b+b'a)+(bb+ba)=(b'+b)(b+a)=b+a$$

by 9, 6-($T3$), ($T2$)-8'-($T1$)-($L3$)-($T1$), 6-($T3$), ($T2$)-9.

Now, as ($T5$) is the same thing as Set I*, 3., Sets I* and II* are equivalent.

If we introduce as in I, the Postulates

10_1. $a+a=a$,

$10'_2$. $(a'+a)+a=a'$

then we obtain the following theorems.

($T6$) The following set of postulates on K characterizes the Boolean lattice:

Set III: 1, 2, 4, 5, 6, 7, 8, 8', 9, 10_1.

($T7$) The following set of postulates on K characterizes the Boolean ring with unity:

Set IV: 1, 2, 4, 5, 6, 7, 8, 8', 9, $10_2'$.

2. A Proof for $(a+b)+c=a+(b+c)$

($L4$) $a(ab)=a(bb)$.

Proof. $a(ab)=(aa)b=a(bb)$ by ($T4$), ($T1$).

($L5$) $a\{(a+b)+c\}=a\{a+(b+c)\}$.

Proof. $a\{(a+b)+c\}=a[\{(a+b)+c\}(c'+c)]$

$$=a[\{(ac'+bc')+cc'\}+\{(ac+bc)+cc\}]$$
$$=a\{(ac'+bc')+cc'\}+[\{a(ac)+a(bc)\}+a(cc)]$$
$$=a\{ac'+(bc'+cc')\}+[a(ac)+\{a(bc)+a(cc)\}]$$
$$=a[\{a+(b+c)\}(c'+c)]=a\{a+(b+c)\}$$

by 9, 6-($T3$)-($T3$), 6-6-6, ($T0$)-8-8-($T0$)-($T5$)-($T5$)-($L4$), 6-6(-$T3$)-($T3$)-6-6, 9.

($L6$) $a'\{(a+b)+c\}=a'\{a+(b+c)\}$.

Proof. $a'\{(a+b)+c\}=(a'a+a'b)+a'c=a'b+a'c$

$$=a'a+(a'b+a'c)=a'\{a+(b+c)\}\cdot$$

by 6-6, 8'-8, 8-8', 6-6.

($T8$) $(a+b)+c=a+(b+c)$.

Proof. $(a+b)+c=(a'+a)\{(a+b)+c\}$

$$=a'\{(a+b)+c\}+a\{(a+b)+c\}$$
$$=a'\{a+(b+c)\}+a\{a+(b+c)\}$$
$$=(a'+a)\{a+(b+c)\}=a+(b+c)$$

by 9-($T2$), ($T3$), ($L6$)-($L5$), ($T3$), ($T2$)-9.

Remark. In Birkhoff,[2)] the Newman algebra K is first decomposed into the direct union of a Boolean lattice K_1 and a (non associative) Boolean ring K_2, and the assooiative law for addition is proved in K_1 and K_2 separately. We notice that our proofs of this law, one given here and the other given in I, are simpler in the sense that they depend neither upon 10_1 nor on $10'_2$.

3. Independence Proofs for Sets II*, III, IV

As said in the introduction, the independence of Set II* follows from that of Set III or of Set IV. We shall denote by $K_i\alpha$ an independence system for Postulate α of K and for i=III, IV, $\alpha=1$, 2, 4, 5, 6, 7, 8, 8′, 9, 10_1, $10'_2$; for example $K_{III}5$ is an independence system of Postulate 5 in K of Set III.

$K_{III}5$:

+	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

×	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	0	c	c

a	a'
0	c
a	b
b	a
c	0

$K_{IV}5$:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

×	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	0	c	c

a	a'
0	c
a	b
b	a
c	0

Here $0=c(ac)\neq a(cc)=a$ in $K_{III}5$ or $K_{IV}5$.

Remark. $K_{III}5$ or $K_{IV}5$ holds for K_I5 or $K_{II}5$ respectively [1): §2].

$K_{IV}8$:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

×	0	a	b	c
0	0	0	0	0
a	0	c	b	a
b	0	b	0	b
c	0	a	b	c

a	a'
0	c
a	b
b	a
c	0

Here $b=0+ab\neq 0$.

The independence of Postulate 1 in each Set III or Set IV is shown by the empty set K. Other systems are easily obtained as two-element systems and to be omitted.

References

1) Wooyenaka, Y.: On Newman Algebra, Proc. Japan Acad., **30**, 170–175 (1954).

2) Birkhoff, G.: Lattice Theory, Am. Math. Soc. Colloquim Publication, **25**, 155–157 (1948).

3) Another proof for ($T2$) $ab=ba$.

Proof. $ab=a\{b(a'+a)\}=a(ba'+ba)=a(ba')+a(ba)=b(a'a)+b(aa)=b(a'a+a)=ba$ by 9, 6, 6, 5, ($T1$)-6, 8′-8.