115. On a Generalization of Groups

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A group can be characterized as a multiplicative system with an operator θ satisfying the following three conditions:

$$\text{I} \qquad (ab)c=a(bc),$$
$$\text{II} \qquad (a^\theta a)b=b,$$
$$\text{III}' \qquad a^\theta a=b^\theta b.$$

Now let us consider about a multiplicative system G with an operator θ satisfying I, II and

$$\text{III} \qquad (ab)^\theta=b^\theta a^\theta.$$

We shall call this a G-system. Then a group is a G-system satisfying $a=a^{\theta\theta}$ for any element a. In this note we shall prove that a G-system is a product of a group and a subsystem consisting of all idempotents.

We shall firstly prove some properties about the operator θ.

1. $a^{\theta\theta\theta}=a^\theta$.

Proof. From II we obtain $a^\theta ab=b$. Multiplying the both sides by $a^{\theta\theta}$ from the left, we have $ab=a^{\theta\theta}b$ by II and $b^\theta a^{\theta\theta\theta}=b^\theta a^\theta$ by III. Multiplying the both sides by $b^{\theta\theta}$ from the left, we have $a^{\theta\theta\theta}=a^\theta$.

2. $ex=x$ and $e^\theta=e$ for $e=a^{\theta\theta}a^\theta$.

Proof. $ex=(a^\theta)^\theta a^\theta x=x$, $e^\theta=(a^{\theta\theta}a^\theta)^\theta=a^{\theta\theta}a^{\theta\theta\theta}=a^{\theta\theta}a^\theta=e$.

3. $a^{\theta\theta}a^\theta=b^{\theta\theta}b^\theta$.

Proof. $b^\theta=(eb)^\theta=b^\theta e^\theta=b^\theta e$, hence $b^{\theta\theta}b^\theta=b^{\theta\theta}b^\theta e=e$.

4. $a^\theta a^{\theta\theta}=a^{\theta\theta}a^\theta$.

Proof. Putting $b=a^\theta$ in 3 we obtain $a^{\theta\theta}a^\theta=a^{\theta\theta\theta}a^{\theta\theta}=a^\theta a^{\theta\theta}$.

5. $xe=x^{\theta\theta}$.

Proof. If we put $y=xe$, then $x^\theta xe=x^\theta y$ and $e=x^\theta y$. Therefore $y=x^{\theta\theta}x^\theta y=x^{\theta\theta}e=x^{\theta\theta}x^\theta x^{\theta\theta}=x^{\theta\theta}$.

6. $e=aa^\theta$.

Proof. $ae=a^{\theta\theta}$ by 5. Multiplying the both sides by a^θ from the right, we have $aea^\theta=a^{\theta\theta}a^\theta=e$. On the other hand, $aea^\theta=a(ea^\theta)=aa^\theta$.

Since θ is an anti-endomorphism of G and the condition III′ holds in G^θ by 3, G^θ is a group. Let $\{C(a)\}$ be the set of classes $C(a)$ of G induced by the anti-endomorphisms θ, where $C(a)$ is the class involving a. Then the set forms a group anti-isomorphic to G^θ.

Theorem 1. *$C(e)$ is a set of all idempotents in G.*

Proof. II implies $a^\theta a^2=a$, therefore $a^\theta a=a$, $a^\theta=(a^\theta a)^\theta=a^\theta a^{\theta\theta}=e$

for an idempotent a. If conversely $a^{\theta}=e$, then $ea^2=a$ by II and $a^2=a$ by 2.

Corollary. $b \in C(a)$ if and only if $a^{\theta}b \in C(e)$.

Lemma 1. If $f \in C(e)$, then $fa=a$ for any element a in G.

Proof. $fa=f^{\theta}fa=a$, since $f^{\theta}=e$.

Lemma 2. $C(a)=aC(e)$.

Proof. $f \in C(e)$ implies $(af)^{\theta}=ea^{\theta}=a^{\theta}$, therefore $aC(e) \subset C(a)$. Conversely $x \in C(a)$ implies, by Corollary of Theorem 1, the existence of f such that $a^{\theta}x=f$, $f \in C(e)$. Then we have $x=a^{\theta\theta}f=aef=af$ by Lemma 1. Therefore $C(a) \subset aC(e)$ and consequently $C(a)=aC(e)$.

Theorem 2. *G is the product $G^{\theta}C(e)$ of the group G^{θ} and the subsystem $C(e)$ consisting of all idempotents. More precisely, the element of G can be uniquely represented as the product of elements of G^{θ} and $C(e)$. The product ab of elements $a=xf$, $b=yg$; $x, y \in G$, $f, g \in C(e)$, is given by $ab=xyg$.*

Proof. Since $a=a^{\theta\theta}a^{\theta}a$ and $a^{\theta}a$ is an idempotent, any element a can be represented in the form $a=xf$. If $a=xf$, $b=yg$, then $ab=xfyg=xyg$ by Lemma 1. Now we prove the uniqueness of the representation. If $a=xf=x'f'$, then multiplying the both sides by e from the right we have $xfe=x'f'e$, $xe=x'e$ by Lemma 1 and $x=x'$, since x, x' are elements in G^{θ}. Multiplying the both sides of $xf=xf'$ by x^{θ} from the left we have $f=f'$ by Lemma 1, since $x^{\theta}x \in C(e)$.

Theorem 3. *The following four conditions are equivalent.*

(1) *There exists an element x in G satisfying $xb^{\theta}=b^{\theta}f$ for any b^{θ} in G^{θ} and any f in $C(e)$.*

(2) *$C(e)$ has only one element.*

(3) *$a=a^{\theta\theta}$ for any element a in G.*

(4) *G is a group.*

Proof. (1)→(2): Multiplying the both sides of $xb^{\theta}=b^{\theta}f$ by $b^{\theta\theta}$ from the right we have an idempotent $xe=b^{\theta}fb^{\theta\theta}$. Then x is an element in $C(e)$, since $e=(xe)^{\theta}=e^{\theta}x^{\theta}=x^{\theta}$. Therefore $b^{\theta}=b^{\theta}f$ by Lemma 1 and $f=b^{\theta\theta}b^{\theta}=e$.

(2)→(3): Since $C(e)$ has only one element, $C(a)$ consists of only one element $ae=a^{\theta\theta}$ and therefore $a=a^{\theta\theta}$.

(3)→(4): (3) implies $G=G^{\theta}$, therefore G is a group.

(4)→(1): (1) follows immediately from $C(e)=e$.

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