145. On the Characterization of the Harmonic Functions

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(Comm. by Z. SUETUNA, M.J.A., Oct. 12, 1954)

§1. **Introduction.** The well-known Green formula for functions of two variables, may be stated as follows:

$$(1)\quad \iint_R \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dxdy+\iint_R v\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big)dxdy=\int_C v\frac{\partial u}{\partial n}\,ds,$$

where $u(x, y)$ and $v(x, y)$ are functions of class C^2 and R is a bounded planar region with boundary C. Then, from (1) we have

Theorem 1. *If u and v are harmonic in R, then*

$$(2)\quad 2\iint_R \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dxdy-\int_C\Big(v\frac{\partial u}{\partial n}+u\frac{\partial v}{\partial n}\Big)ds=0.$$

In §2, we shall prove a theorem which is a sort of inverse of Theorem I. For the proof, we use the method due to Beckenbach [1]. On the other hand it is known that

Theorem 2. *If $u(x, y)$ is harmonic in a planar domain R, then for any closed circle $C(x, y; r)$ contained in R.*

$$(3)\quad \frac{1}{2\pi}\int_0^{2\pi} u(x+r\cos\theta,\ y+r\sin\theta)\,d\theta -\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^{r} u(x+\rho\cos\theta,\ y+\rho\sin\theta)\,\rho d\rho d\theta=0.$$

Further Levi [2] and Tonelli [3] proved that if $u(x, y)$ is continuous in R and (3) holds for any closed circle C contained in R, then $u(x, y)$ is harmonic in R.

We prove a similar theorem in §3.

§2. **Lemma 1** (Saks [4]). *If $u(x, y)$ belongs to the class C^1 and for any closed circle $C(x, y; r)$ contained in D*

$$\int_0^{2\pi}\frac{\partial u}{\partial n}\,r d\theta=o(r^2),^{1)}$$

then, $u(x, y)$ is harmonic in D.

As an inverse of Theorem 1, we prove

Theorem I. *If $u(x, y)$ and $v(x, y)$ belong to the class C^1 in a*

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1) $\phi(r)=o(r^\alpha)$ means that $\lim\limits_{r\to 0}\frac{\phi(r)}{r^\alpha}=0$.

planar region R, *and* $v(x, y)$ *is harmonic and* $\neq 0$ *in* R, *and further if for any closed circle* $C(x, y; r)$ *contained in* R

$$(4)\qquad 2\int_0^{2\pi}\int_0^{r}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\rho d\rho d\theta-\int_0^{2\pi}\left(v\frac{\partial u}{\partial n}+u\frac{\partial v}{\partial n}\right)rd\theta=o(r^2),$$

then $u(x, y)$ *is harmonic in* R.

Proof. Since u and v belong to class C^1 in R, we have for each point (x_0, y_0) in R,

$$(5)\qquad \begin{cases} u(x, y)=u(x_0, y_0)+a_1 r\cos\theta+b_1 r\sin\theta+o(r) \\ v(x, y)=v(x_0, y_0)+a_2 r\cos\theta+b_2 r\sin\theta+o(r), \end{cases}$$

where a_1, b_1, a_2, b_2 denote the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at the point (x_0, y_0) and $x-x_0=r\cos\theta$, $y-y_0=r\sin\theta$.

Therefore,

$$(6)\qquad \begin{cases} \frac{\partial u}{\partial x}=a_1+o(1),\ \frac{\partial u}{\partial y}=b_1+o(1),\ \frac{\partial v}{\partial x}=a_2+o(1), \\ \frac{\partial v}{\partial y}=b_2+o(1) \\ \frac{\partial u}{\partial n}=a_1\cos\theta+b_1\sin\theta+o(1),\quad \frac{\partial v}{\partial n}=a_2\cos\theta+b_2\sin\theta+o(1). \end{cases}$$

By (6) and (7), we get the following relations:

$$u\frac{\partial v}{\partial n}=u(x_0, y_0)\frac{\partial v}{\partial n}+a_1a_2r\cos^2\theta+(a_2b_1+a_1b_2)r\sin\theta\cos\theta$$
$$+b_1b_2r\sin^2\theta+o(r),$$
$$v\frac{\partial u}{\partial n}=v(x_0, y_0)\frac{\partial u}{\partial n}+a_1a_2r\cos^2\theta+(a_1b_2+b_1a_2)r\sin\theta\cos\theta$$
$$+b_1b_2r\sin^2\theta+o(r),$$

and hence

$$(7)\qquad \int_0^{2\pi}\left(u\frac{\partial v}{\partial n}+v\frac{\partial u}{\partial n}\right)rd\theta=u(x_0, y_0)\int_0^{2\pi}\frac{\partial v}{\partial n}rd\theta+v(x_0, y_0)\int_0^{2\pi}\frac{\partial u}{\partial n}rd\theta$$
$$+2\pi(a_1a_2+b_1b_2)r^2+o(r^2).$$

Since $v(x, y)$ is harmonic in R,

$$8)\qquad \int_0^{2\pi}\frac{\partial v}{\partial n}rd\theta=0.$$

By (7) and (8), we get

$$(9)\qquad \int_0^{2\pi}\left(u\frac{\partial v}{\partial n}+v\frac{\partial u}{\partial n}\right)rd\theta=v(x_0, y_0)\int_0^{2\pi}\frac{\partial u}{\partial n}rd\theta$$
$$+2\pi(a_1a_2+b_1b_2)r^2+o(r^2).$$

On the other hand

$$(10)\qquad 2\int_0^{2\pi}\int_0^r\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]\rho d\rho d\theta=2\pi r^2(a_1a_2+b_1b_2)+o(r^2).$$

From (4), (9) and (10), we get

$$2\int_0^{2\pi}\int_0^r\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial n}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]\rho d\rho d\theta-\int_0^{2\pi}\left(v\frac{\partial u}{\partial n}+u\frac{\partial v}{\partial n}\right)rd\theta$$
$$=-v(x_0,y_0)\int_0^{2\pi}\frac{\partial u}{\partial n}rd\theta+o(r^2).$$

Since $v(x_0, y_0)\neq 0$, we see immediately that for any closed circle $C(x_0, y_0; r)$ contained in R,

$$\int_0^{2\pi}\frac{\partial u}{\partial n}rd\theta=o(r^2).$$

Therefore, applying Lemma 1, we see that $u(x, y)$ is harmonic in R, q.e.d.

Lemma 2 (Beckenbach [1]). *Let $u(x, y)$ be a function of class C^1 in R, and suppose that for any point (x_0, y_0) in R, one of the following conditions is satisfied:*

(i) *There exists a neighborhood of (x_0, y_0) in which $u(x, y)$ is harmonic,*

(ii) *$u(x_0, y_0)=0$ and $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial y^2}$ are summable in R, then $u(x, y)$ is harmonic in R.*

This Lemma and Theorem I give the following.

Theorem II. *Let $u(x, y)$ be a function of class C^1 in R, and $v(x, y)$ be harmonic in R. And further the following two conditions are satisfied:*

(i) *if $v(x, y)=0$, then $u(x, y)=0$ and moreover $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial y^2}$ are summable in R,*

(ii) *the relation* (4) *holds for any closed circle $C(x, y; r)$ contained in R.*

Then $u(x, y)$ is harmonic in R.

Similarly we can prove the following.

Theorem III. *Let $u(x, y)$ be a function of class C^1 in R, and $u(x, y)>0$.*

If for any closed circle $C(x, y; r)$ contained in R,

$$(11)\qquad \lambda u(x,y)^{\lambda-1}\int_0^{2\pi}\int_0^r\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\rho d\rho d\theta$$
$$-\int_0^{2\pi}u^\lambda\frac{\partial u}{\partial n}rd\theta=o(r^2)\quad(\lambda\geqq 1),$$

then $u(x, y)$ is harmonic in R.

When $\lambda=1$, this theorem reduces to a theorem of Beckenbach [1].

§3. We shall prove a theorem in the direction of the Levi-Tonelli theorem.

Theorem IV. *Let $u(x, y)$ be a function of class C^1 in R, and suppose that for any closed circle $C(x, y; r)$ contained in R the following conditions are satisfied:*

$$\frac{1}{2\pi}\int_0^{2\pi} u(x+r\cos\theta, y+r\sin\theta)d\theta-\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r u(x+\rho\cos\theta, y+\rho\sin\theta)\rho d\rho d\theta=o(r^2), \tag{12}$$

$$\int_0^{2\pi}\operatorname*{Max}_{0\le\rho\le r}[\{u_x(x+r\cos\theta, y+r\sin\theta)-u_x(x+\rho\cos\theta, y+\rho\sin\theta)\}\cos\theta+\{u_y(x+r\cos\theta, y+r\sin\theta)-u_y(x+\rho\cos\theta, y+\rho\sin\theta)\}\sin\theta]d\theta=o(r) \tag{13}$$

then $u(x, y)$ is harmonic in R.

Proof. Let us denote by Δ the left-side of (12). Then

$$\Delta=\frac{1}{2\pi}\int_0^{2\pi} d\theta\frac{2}{r^2}\int_0^r\{u(x+r\cos\theta, y+r\sin\theta)-u(x+\rho\cos\theta, y+\rho\sin\theta)\}\rho d\rho. \tag{14}$$

By the mean value theorem, we get

$$\begin{aligned}
&u(x+r\cos\theta, y+r\sin\theta)-u(x+\rho\cos\theta, y+\rho\sin\theta)\\
&=u_x(x+\rho'\cos\theta, y+\rho'\sin\theta)(r-\rho)\cos\theta+u_y(x+\rho'\cos\theta, y+\rho'\sin\theta)(r-\rho)\sin\theta\\
&=\{u_x(x+r\cos\theta, y+r\sin\theta)\cos\theta+u_y(x+r\cos\theta, y+r\sin\theta)\sin\theta\}(r-\rho)\\
&\quad+(r-\rho)\{[u_x(x+\rho'\cos\theta, y+\rho'\sin\theta)-u_x(x+r\cos\theta, y+r\sin\theta)]\cos\theta\\
&\quad+[u_y(x+\rho'\cos\theta, y+\rho'\sin\theta)-u_y(x+r\cos\theta, y+r\sin\theta)]\sin\theta\}
\end{aligned}$$

$$= P_1+P_2, \text{ say,} \tag{15}$$

where $0<\rho<\rho'<r$.

Now we get

$$\begin{aligned}
I_1&=\frac{1}{2\pi}\int_0^{2\pi} d\theta\left\{\frac{2}{r^2}\int_0^r P^1\rho d\rho\right\}\\
&=\frac{1}{2\pi}\int_0^{2\pi} d\theta\left\{\frac{2}{r^2}\int_0^r\frac{\partial u(x+r\cos\theta, y+r\sin\theta)}{\partial n}\rho(r-\rho)d\rho\right\}
\end{aligned}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} d\theta\left\{\frac{2}{r^2}\frac{\partial u}{\partial n}\left(\frac{r^3}{2}-\frac{r^3}{3}\right)\right\}=\frac{1}{6\pi}\int_0^{2\pi}\frac{\partial u}{\partial n}rd\theta, \tag{16}$$

and by (13)

$$I_2=\frac{1}{2\pi}\int_0^{2\pi}d\theta\left\{\frac{2}{r^2}\int_0^r P_2\,\rho d\rho\right\}$$

$$=\frac{2}{r^2}\int_0^r(r-\rho)\rho d\rho\int_0^{2\pi}[\{u_x(x+\rho'\cos\theta,\ y+\rho'\sin\theta)-u_x$$

$$(x+r\cos\theta,\ y+r\sin\theta)\}\cos\theta+\{u_y(x+\rho'\cos\theta,\ y+\rho'\sin\theta)$$

$$-u_y(x+r\cos\theta,\ y+r\sin\theta)\}\sin\theta]\,d\theta$$

(17) $$=\frac{2}{r^2}\int_0^r o(r)(r-\rho)\rho d\rho=o(r)\frac{2}{r^2}\cdot\frac{r^3}{6}=o(r^2).$$

By the assumption (12),

(18) $$\Delta=o(r^2).$$

Combining the estimations (14), (15), (16), (17) and (18), we get

$$\Delta=\frac{1}{6\pi}\int_0^{2\pi}\frac{\partial u}{\partial n}r d\theta+o(r^2)=o(r^2).$$

Therefore

$$\int_0^{2\pi}\frac{\partial u}{\partial n}r d\theta=o(r^2).$$

By Lemma 1, we see that $u(x, y)$ is harmonic in R, q.e.d. Finally, we conjecture the following proposition:

If $u(x, y)$ belongs to class C in R, and for any closed circle $C(x, y; r)$ contained in R the relation (12) holds, then $u(x, y)$ is harmonic in R.

References

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