167. A Note on Strongly (C, α)-ergodic Semi-Group of Operators

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Let $\{T(\xi): 0 < \xi < \infty\}$ be a semi-group of operators satisfying the following assumptions:

(i) *For each* ξ, $0<\xi<\infty$, $T(\xi)$ *is a bounded linear operator from a complex Banach space* X *into itself and*

(1) $$T(\xi+\eta)=T(\xi)T(\eta).$$

(ii) $T(\xi)$ *is strongly measurable in* $(0, \infty)$.

(iii) $$\int_0^1 \|T(\xi)x\|\, d\xi < \infty \qquad \textit{for each } x \in X.$$

We may further assume without loss of generality that

(iv) $\|T(\xi)\|$ *is bounded at* $\xi=\infty$.

If $T(\xi)$ satisfies the condition

(v) $$\lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda\xi}T(\xi)x d\xi = x \qquad \textit{for each } x \in X,$$

then $T(\xi)$ is said to be strongly *Abel-ergodic* to the identity at zero. If, instead of (v), $T(\xi)$ satisfies the stronger condition

(v′) $$\lim_{\xi\to 0} \alpha\xi^{-\alpha} \int_0^\xi (\xi-\eta)^{\alpha-1}T(\eta)x d\eta = x \qquad \textit{for each } x \in X,$$

then $T(\xi)$ is said to be strongly *(C, α)-ergodic* to the identity at zero.

Recently R. S. Phillips [1] and the present author [3] have independently proved the following

Theorem 1. *A necessary and sufficient condition that a semi-group of operators strongly Abel-ergodic to the identity at zero be of operators strongly* (C, 1)*-ergodic to the identity at zero is that there exists a positive number* M *such that*

(2) $$\sup_{k\geq 1,\lambda>0} \left\| \frac{1}{k}\sum_{i=1}^{k}[\lambda R(\lambda; A)]^i \right\| \leqq M.$$

In this note we shall give a generalization of Theorem 1 which is stated as follows:

Theorem 2. *Let* α *be a positive integer. A necessary and sufficient condition that a semi-group of operators strongly Abel-ergodic to the identity at zero be of operators strongly* (C, α)*-ergodic to the identity at zero is that there exists a positive number* M *such that*

(3) $$\sup_{\lambda>0,k\geq\alpha} \left\| \frac{\alpha}{k(k-1)\cdots(k-\alpha+1)} \sum_{i=1}^{k-\alpha+1} \frac{(k-i)!}{(k-\alpha+1-i)!}[\lambda R(\lambda; A)]^i \right\| \leqq M.$$

We denote the infinitesimal generator of $T(\xi)$ by A and the domain of A by $D(A)$. If $T(\xi)$ is a semi-group of operators strongly Abel-ergodic to the identity at zero, then the following properties are known [1], [3].

(a) *The operator* $R(\lambda; A)$ *defined by*

$$(4) \qquad R(\lambda; A)x=\int_0^\infty e^{-\lambda\xi}T(\xi)x d\xi \qquad \textit{for each } \lambda>0,$$

is a bounded linear operator from X into itself.

(b) $$R(\lambda; A)(\lambda-A)x=x \qquad \textit{for each } x\in D(A),$$
$$(\lambda-A)R(\lambda; A)x=x$$

for each x such that $\lim_{\xi\to 0}\xi^{-1}\int_0^\xi T(\eta)x d\eta=x$.

(c) $D(A)$ *is a dense subset in* X.

We remark that the (C, α)-ergodicity implies the Abel-ergodicity.

Proof of Theorem 2. By the conditions (b) and (c) or (4), we obtain the resolvent equation

$$(5) \qquad R(\lambda; A)-R(\mu; A)=-(\lambda-\mu)R(\lambda; A)R(\mu; A)$$

for positive numbers λ and μ, and then

$$(6) \qquad [\lambda R(\lambda; A)]^k x=\frac{\lambda^k}{(k-1)!}\int_0^\infty e^{-\lambda\tau}\tau^{k-1}T(\tau)x d\tau, \qquad k=1, 2, \dots,$$

from (4) and (5). For any positive integer α, we get

$$\frac{\lambda^{k+1}}{k!}\int_0^\infty e^{-\lambda\xi}\xi^k\left[\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}T(\tau)x d\tau\right]d\xi$$
$$=\frac{\lambda^{k+1}}{k!}\int_0^\infty T(\tau)x\left[\int_\tau^\infty \alpha e^{-\lambda\xi}\xi^{k-\alpha}(\xi-\tau)^{\alpha-1}d\xi\right]d\tau$$
$$=\frac{\lambda^{k+1}}{k!}\alpha\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\int_0^\infty \tau^i T(\tau)x\left[\int_\tau^\infty \xi^{k-1-i}e^{-\lambda\xi}d\xi\right]d\tau$$
$$=\frac{\lambda^{k+1}}{k!}\alpha\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\left[\frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}\tau^{k-1}T(\tau)x d\tau\right.$$
$$+\frac{k-1-i}{\lambda^2}\int_0^\infty e^{-\lambda\tau}\tau^{k-2}T(\tau)x d\tau+\cdots$$
$$\left.+\frac{(k-1-i)!}{\lambda^{k-i}}\int_0^\infty e^{-\lambda\tau}\tau^i T(\tau)x d\tau\right],$$

where $k\geqq\alpha$. Therefore we have by (6)

$$\frac{\lambda^{k+1}}{k!}\int_0^\infty e^{-\lambda\xi}\xi^k\left[\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}T(\tau)x d\tau\right]d\xi$$
$$=\frac{\alpha}{k!}\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\Big\{(k-1)![\lambda R(\lambda; A)]^k$$
$$+(k-1-i)(k-2)![\lambda R(\lambda; A)]^{k-1}+\cdots$$
$$+(k-1-i)!\,i!\,[\lambda R(\lambda; A)]^{i+1}\Big\}x$$

$$=\frac{\alpha}{k(k-1)\cdots(k-\alpha+1)}\left\{\frac{(k-1)!}{(k-\alpha)!}[\lambda R(\lambda;\ A)]^k\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\right.$$

$$+\frac{(k-2)!}{(k-\alpha)!}[\lambda R(\lambda;\ A)]^{k-1}\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(k-1-i)+\cdots$$

$$+\frac{(k-\alpha)!}{(k-\alpha)!}[\lambda R(\lambda;\ A)]^{k-\alpha+1}\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(k-1-i)\cdots(k-\alpha+1-i)$$

$$+\cdots+\frac{(\alpha-1)!}{(k-\alpha)!}[\lambda R(\lambda;\ A)]^{\alpha}\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(k-1-i)\cdots(\alpha-i)$$

$$+\frac{(\alpha-2)!}{(k-\alpha)!}[\lambda R(\lambda;\ A)]^{\alpha-1}\sum_{i=0}^{\alpha-2}(-1)^i\binom{\alpha-1}{i}(k-1-i)\cdots(\alpha-1-i)+\cdots$$

$$+\frac{1\,!}{(k-\alpha)!}[\lambda R(\lambda;\ A)]^{2}\sum_{i=0}^{1}(-1)^i\binom{\alpha-1}{i}(k-1-i)\cdots(2-i)$$

$$\left.+\frac{(k-1)!}{(k-\alpha)!}[\lambda R(\lambda;\ A)]\right\}x.$$

Let us put

$$F(x)=x^{k-\alpha}(x-1)^{\alpha-1}=\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}x^{k-1-i}.$$

Then it is obvious by the Leibniz formula that

$$F^{(j)}(1)=\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(k-1-i)\cdots(k-j-i)$$

$$=\begin{cases}0 & \text{for } j\leqq\alpha-2,\\ (\alpha-1)! & \text{for } j=\alpha-1,\\ \binom{j}{j-\alpha+1}(\alpha-1)!(k-\alpha)\cdots(k-j) & \text{for } k-1\geqq j\geqq\alpha.\end{cases}$$

Thus we obtain

$$(7)\qquad \frac{\lambda^{k+1}}{k\,!}\int_0^\infty e^{-\lambda\xi}\xi^k\left[\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}T(\tau)xd\tau\right]d\xi$$

$$=\frac{\alpha}{k(k-1)\cdots(k-\alpha+1)}\sum_{i=1}^{k-\alpha+1}\frac{(k-i)!}{(k-\alpha+1-i)!}[\lambda R(\lambda;\ A)]^i x\quad\text{for } k\geqq\alpha.$$

We first prove the necessity. From the strong (C, α)-ergodicity and the condition (iv), there exists a positive number M such that

$$(8)\qquad \left\|\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}T(\tau)xd\tau\right\|\leqq M\,\|x\|\qquad\text{for } \xi>0.$$

Thus we have the relation (3) by (7) and (8).

On the other hand, using the well-known theorem that if $f(\xi)$ is a bounded continuous function and $k/\lambda\to\eta$ $(\lambda=\lambda(k)\to\infty,\ k\to\infty)$ then

$$\frac{\lambda^{k+1}}{k\,!}\int_0^\infty e^{-\lambda\xi}\xi^k f(\xi)d\xi\to f(\eta),$$

we have by (7)

$$\liminf \left\| \frac{\alpha}{k(k-1)\cdots(k-\alpha+1)} \sum_{i=1}^{k-\alpha+1} \frac{(k-i)!}{(k-\alpha+1-i)!} [\lambda R(\lambda; A)]^i \right\| \geqq \left\| \alpha\eta^{-\alpha} \int_0^{\eta} (\eta-\tau)^{\alpha-1} T(\tau) x d\tau \right\|.$$

Therefore we get by (3)

$$\sup_{\xi>0} \left\| \alpha\xi^{-\alpha} \int_0^{\xi} (\xi-\tau)^{\alpha-1} T(\tau) x d\tau \right\| \leqq M \|x\|.$$

Thus the sufficiency follows from the following theorem [2; Theorem 2].

Theorem. *If the semi-group of operators* $\{T(\xi);\ 0<\xi<\infty\}$ *satisfies the conditions* (*i*), (*ii*) *and* (*iii*), *if* $T(\xi)$ *is strongly Abel-ergodic at zero, and further if*

$$\| \alpha\xi^{-\alpha} \int_0^{\xi} (\xi-\tau)^{\alpha-1} T(\tau) x d\tau \| \leqq M \|x\| \qquad \text{for } 0<\xi<1,$$

then $T(\xi)$ *is strongly* (C, α)*-ergodic at zero.*

Remark 1. Necessary and sufficient conditions in order that a operator A generates a semi-group of operators strongly Abel-ergodic to the identity at zero are given by R. S. Phillips [1] and the present author [3]. Thus we can obtain, by Theorem 2, necessary and sufficient conditions that a given operator A generates a semi-group of operators strongly (C, α)-ergodic (α=positive integer) to the identity at zero.

Remark 2. We note that the sufficiency can be proved as follows. We have $\lim_{\xi\to 0} T(\xi)x = x$ for $x \in D(A)$ and a fortiori

$$\lim_{\xi\to 0} \alpha\xi^{-\alpha} \int_0^{\xi} (\xi-\tau)^{\alpha-1} T(\tau) x d\tau = x.$$

Since $D(A)$ is a dense set in X, we see by the Banach-Steinhaus theorem that the latter relation is true for all $x \in X$.

References

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