14. *On a Characteristic Property of Completely Normal Spaces*

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Let H be a topological space. We shall consider the following condition:

(V) *For any two subsets* X_1 *and* X_2 *of* H, *there exist two closed sets* H_1 *and* H_2 *such that*

(1) $$H=H_1 \smile H_2$$

(2) $$H_1 \frown H_2 \frown (\overline{X}_1 \smile \overline{X}_2)=\overline{X}_1 \frown \overline{X}_2$$

(3) $$\overline{X}_i \subset H_i, \qquad i=1, 2.$$

Here the bar means the closure operation.

As is well known, the metrizability of H implies (V) and (V) implies the normality of H. However it seems that a closer relation between (V) and the separation axioms has not been given in the literature.[1]

The object of this note is to show that a topological space H is completely normal if and only if it satisfies the condition (V).

1. First we shall prove

Lemma 1. The conditions (1), (2), and (3) imply (1), (3), and (2′) below:

(2′) $$H_i \frown (\overline{X}_1 \smile \overline{X}_2) \subset \overline{X}_i, \qquad i=1, 2,$$

and conversely.

Proof. It is obvious that (1), (2′), and (3) imply (2); we have only to prove that (1), (2), and (3) imply (2′). Since $H_1 \frown H_2 \frown (\overline{X}_1 \smile \overline{X}_2) \frown \overline{X}_2=\overline{X}_1 \frown \overline{X}_2 \frown \overline{X}_2$ by (2), we have $H_1 \frown \overline{X}_2=\overline{X}_1 \frown \overline{X}_2$ by (3). Therefore $(H_1 \frown \overline{X}_2) \smile \overline{X}_1=(\overline{X}_1 \frown \overline{X}_2) \smile \overline{X}_1$, and hence $H_1 \frown (\overline{X}_1 \smile \overline{X}_2) \subset \overline{X}_1$. Similarly we have $H_2 \frown (\overline{X}_1 \smile \overline{X}_2) \subset \overline{X}_2$.

Next we shall recall the definition of completely normal spaces; a topological space H is said to be completely normal if the following condition is satisfied:

For any two subsets Y_1 and Y_2 of H such that

(4) $$\overline{Y}_1 \frown Y_2 = Y_1 \frown \overline{Y}_2 = 0,$$

1) Cf. A. D. Wallace: Dimensional types, Bull. Amer. Math. Soc., **51**, 679–681 (1945). Indeed, he said in his paper "the validity of (V) is a well-known property of metric spaces but we have no reference to its formulation in the literature as an axiom".

there exist two open sets K_1 and K_2 of H with the following properties:

(5) $$K_1 \frown K_2 = 0$$

(6) $$Y_i \subset K_i, \qquad i=1, 2.$$

Now we shall prove

Theorem 1. *A necessary and sufficient condition that a topological space H be completely normal is that H satisfy the condition (V).*

Proof. Sufficiency. Assume that (V) is satisfied. Let Y_1 and Y_2 be any two subsets of H satisfying (4). Then we can find two closed sets H_1 and H_2 which satisfy (1), (2), and (3) with X_1 and X_2 replaced by Y_1 and Y_2. Let us set $K_1 = H - H_2$ and $K_2 = H - H_1$. Then K_i are open and the validity of (5) is obvious from (1). Next we shall show that (6) holds. Suppose that there exists a point x which is contained in Y_1 but not in K_1. Then x is contained in H_2. Since $x \in Y_1 \subset H_1$ by (3), we have $x \in H_1 \frown H_2 \frown (\overline{Y}_1 \smile \overline{Y}_2) = \overline{Y}_1 \frown \overline{Y}_2$. Thus $x \in Y_1 \frown \overline{Y}_2$, this contradicts to (4). Hence we have $Y_i \subset K_i$, $i=1, 2$. Therefore H is completely normal.

Necessity. Let X_1 and X_2 be any two subsets of H. We put $Y_1 = \overline{X}_2 - \overline{X}_1$ and $Y_2 = \overline{X}_1 - \overline{X}_2$. Then $\overline{Y}_1 \subset \overline{X}_2$ and $\overline{Y}_2 \subset \overline{X}_1$, and hence $\overline{Y}_i \frown Y_j \subset \overline{X}_j \frown (\overline{X}_i - \overline{X}_j) = 0$, $i, j = 1, 2$. Thus (4) is fulfilled. Assume that H is completely normal. Then we can find two open sets K_1 and K_2 satisfying (5) and (6). Let us put

(7) $$K_i' = K_i \frown (H - \overline{X}_i), \quad H_i = H - K_i', \qquad i=1, 2.$$

It is obvious that H_1 and H_2 are closed and satisfy (1) and (3). We shall show that (2′) holds also. Since $Y_1 = \overline{X}_2 - \overline{X}_1 \subset K_1$, we have $H - K_1 \subset (H - \overline{X}_2) \smile \overline{X}_1$, and hence $H_1 = H - K_1' = (H - K_1) \smile \overline{X}_1 \subset (H - \overline{X}_2) \smile \overline{X}_1$ by (7). Therefore $H_1 \frown (\overline{X}_1 \smile \overline{X}_2) \subset ((H - \overline{X}_2) \smile \overline{X}_1) \frown (\overline{X}_1 \smile \overline{X}_2) = \overline{X}_1$. Similarly we obtain $H_2 \frown (\overline{X}_1 \smile \overline{X}_2) \subset \overline{X}_2$. This shows that if H is completely normal, then H satisfies the condition (V).

Thus Theorem 1 is completely proved.

2. We can extend Theorem 1 as follows:

Theorem 2. *If a topological space H is completely normal, then for any n subsets $X_1, X_2, \dots, X_n$ of H, there exist n closed sets H_1, $H_2, \dots, H_n$ such that*

(1_n) $$H = H_1 \smile H_2 \smile \dots \smile H_n$$

(2_n) $$H_i \frown H_j \frown (\overline{X}_i \smile \overline{X}_j) = \overline{X}_i \frown \overline{X}_j$$

(3_n) $$\overline{X}_i \subset H_i, \qquad i=1, 2, \dots, n.$$

Proof. We shall prove our theorem by induction on n. In case $n=2$, Theorem 2 is true by Theorem 1. Assume that Theorem 2 holds for n. Let $X_1, X_2, \dots, X_{n+1}$ be $n+1$ subsets of H. Then we can find n closed sets $H_1, H_2, \dots, H_{n-1}$, and H' satisfying the conditions of Theorem 2 for n given sets $X_1, X_2, \dots, X_{n-1}$ and $X' = X_n \smile X_{n+1}$. Since H' is completely normal, by Theorem 1 we can select two closed subsets H_n and H_{n+1} of H' which satisfy (V) for the two given subsets X_n and X_{n+1} of H'. We shall show that these $H_1, H_2, \dots, H_{n+1}$ fulfill the conditions of Theorem 2 for $X_1, X_2, \dots, X_{n+1}$. The conditions (1_n) and (3_n) hold trivially. To prove the equality $H_i \frown H_n \frown (\overline{X}_i \smile \overline{X}_n) = \overline{X}_i \frown \overline{X}_n$ for $i<n$, let $x \in H_i \frown H_n \frown (\overline{X}_i \smile \overline{X}_n)$. Then we have $x \in H_i \frown H' \frown (\overline{X}_i \smile \overline{X_n \smile X_{n+1}}) = \overline{X}_i \frown \overline{X_n \smile X_{n+1}}$. By Lemma 1 we obtain $x \in H_n \frown (\overline{X_n \smile X_{n+1}}) \subset \overline{X}_n$. Therefore $x \in \overline{X}_i \frown \overline{X}_n$ and hence $H_i \frown H_n \frown (\overline{X}_i \smile \overline{X}_n) \subset \overline{X}_i \frown \overline{X}_n$. Since it is obvious that $H_i \frown H_n \frown (\overline{X}_i \smile \overline{X}_n) \supset \overline{X}_i \frown \overline{X}_n$, we have $H_i \frown H_n \frown (\overline{X}_i \smile \overline{X}_n) = \overline{X}_i \frown \overline{X}_n$. Since the above argument holds equally for H_{n+1}, the proof is completed.

3. In Theorem 2 the number n is, of course, assumed to be finite. For an infinite number of sets $X_1, X_2, \dots$, a similar proposition does not hold in general, as is shown by the following example.

Let H be Euclidian space of dimension 1, and $X_1, X_2, \dots$ be an infinite family of all the subsets X_i of H such that X_i consists of only one rational point. Then there is no family of closed sets $H_1, H_2, \dots$ of H with the following property:

$$(\mathrm{V}^*) \qquad \begin{cases} H = H_1 \smile H_2 \smile \dots \\ H_i \frown H_j \frown (\overline{X}_i \smile \overline{X}_j) = \overline{X}_i \frown \overline{X}_j \\ \overline{X}_i \subset H_i, \qquad i=1, 2, \dots . \end{cases}$$

In this case H is, of course, a completely normal topological space.

Proof. Suppose that there is a desired family $\{H_1, H_2, \dots\}$. Since $H_i \frown (\overline{X}_i \smile \overline{X}_j) \subset \overline{X}_i$, the number of rational points contained in H_i is one. The set R_0 of all irrational points is equal to $\smile \{H_i - X_i\} = \smile \{H_i \frown (H - X_i)\}$ and hence R_0 is an F_σ-set, for H is closed and $H - X_i$ is open. Since R_0 is a G_δ-set, it is F_σ and G_δ at the same time, that is, R_0 is a developable set in the sense of C. Kuratowski. On the other hand R_0 is a frontier set (ensemble frontier, i.e. $\overline{H - R_0} = H$) and hence R_0 is non dense (i.e. $\overline{H - \overline{R}_0} = H$). This is a contradiction.

4. We have just obtained the consequence that even in Euclidian space of dimension 1, the property (V^*) does not hold in general. So we are led to the consideration of the case where $\{X_1, X_2, \dots\}$ is locally finite.

Theorem 3. *If a topological space H is completely normal and fully normal, then for any locally finite family of subsets $X_1, X_2, \dots$ of H, there exists a family of closed sets $H_1, H_2, \dots$ with the property (V^*).*

Proof. Since the family $\{X_\alpha\}$ is locally finite, the family $\{\overline{X}_\alpha\}$ is also locally finite. So we assume X_α to be closed and shall prove this theorem for the family of closed sets $\{X_\alpha\}$. For any point x of H, there exists a neighbourhood U of x which intersects only a finite number of the sets of $\{X_\alpha\}$. There corresponds a neighbourhood V of x such that $U \supset \overline{V} \supset V$, for H is regular. Thus we obtain an open covering $\{V\}$ such that the closure of any open set of $\{V\}$ intersects only a finite number of sets of $\{X_\alpha\}$. Since H is fully normal, we can find a locally finite open covering $\{W_\beta\}$ which is a refinement of $\{V\}$. Then $\{\overline{W}_\beta\}$ is a locally finite closed covering which is a refinement of $\{\overline{V}\}$. For each β, let $\{X_i;\ i \in \Gamma_{(\beta)}\}$ be the family of the sets X_i intersecting $\overline{W}_\beta$ (here $\Gamma_{(\beta)}$ is finite). For $\{X_i \frown \overline{W}_\beta;\ i \in \Gamma_{(\beta)}\}$ we can obtain $\{H_{i\beta};\ i \in \Gamma_{(\beta)}\}$ such that

$$\overline{W}_\beta = \smile \{H_{i\beta};\ i \in \Gamma_{(\beta)}\},$$

$$H_{i\beta} \frown H_{j\beta} \frown ((X_i \frown \overline{W}_\beta) \smile (X_j \frown \overline{W}_\beta)) = (X_i \frown \overline{W}_\beta) \frown (X_j \frown \overline{W}_\beta),$$

$$X_i \frown \overline{W}_\beta \subset H_{i\beta},$$

by Theorem 2. Let us set $H_i = \underset{\beta}{\smile} \{H_{i\beta}\}$ where the sum extends over β such that $i \in \Gamma_{(\beta)}$. Then H_i is clearly closed. It is obvious that $X_i \subset H_i$ and $H = \smile \{H_\alpha\}$. We shall show $H_i \frown (X_i \smile X_j) \subset X_i$. Let x be any point of $H_i \frown (X_i \smile X_j)$. Then for some β, $x \in H_{i\beta}$. Since $H_{i\beta} \subset \overline{W}_\beta$, $x \in H_{i\beta} \frown ((X_i \frown \overline{W}_\beta) \smile (X_j \frown \overline{W}_\beta))$ and hence by Lemma 1 $x \in (X_i \frown \overline{W}_\beta)$ and consequently $x \in X_i$. Therefore $H_i \frown (X_i \smile X_j) \subset X_i$. The proof is completed.